AN

ELEMENTARY COURSE

OF

MATHEMATICS

FOR THE USE OF THE

ROYAL MILITARY ACADEMY,

AND FOR STUDENTS IN GENERAL.

BY

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PREFACE TO PART III.

This Part of the Course of Mathematics for the Royal Military Academy contains a Supplement to the first six books of Euclid; the Elements of the Geometry of Planes and of Solids; the elementary principles of Descriptive Geometry; and the principles of Isometric Perspective.

It was originally my intention that this Part should contain the Elements of Plane Geometry with the Geometry of Planes and of Solids, but having been called upon to furnish a short treatise on the elementary principles of Descriptive Geometry, adapted to the wants of this Institution, I was desirous of avoiding the delay which would arise from commencing with Plane Geometry. When the greater portion of this Part of the Course was printed, and had for some time been in use in the Academy, a new edition of Euclid's elements, by Mr. Robert Potts, M.A. of Trinity College, Cambridge, which is likely to supersede most others, to the extent, at least, of the six books, was published. From the manner of arranging the demonstration, this edition has the advantages of the symbolical form, and it is at the same time free from the manifold objections to which that form is open. The duodecimo edition of this work, comprising only the first six books of Euclid, with Deductions from them, having been introduced

at this institution as a text book, now renders any other treatise on Plane Geometry unnecessary in our Course of Mathematics. It has however been found necessary to have a Supplement to these elements, in order to establish the principles on which problems on the mensuration of the circle depend, and this has been made to precede the Geometry of Planes.

Besides the propositions in the eleventh book of Euclid, there are several propositions on straight lines and planes which are of great use, particularly as properties to refer to in problems on projections and Descriptive Geometry. These have been added in the Geometry of Planes.

To the Geometry of Solids we must refer for all the principles requisite for obtaining rules for the mensuration of solids. There is here little difficulty until we come to the pyramid, the cylinder, the cone and the sphere. It is true that the Gordian knot which is here presented may be cut by adopting the method of Cavalieri, but besides that this method is erroneous in principle, no conclusion can be drawn from it in the case of the simplest solids, without, in fact, assuming the very thing to be proved: it ought therefore never to be had recourse to in sound elementary instruction. The method which, after much reflection on the subject, I considered the simplest, and at the same time the most satisfactory, differed but little from that adopted by Playfair in the last ten propositions in the third book of the Supplement to his Elements of Geometry. I have therefore here followed his method though I have not always adhered to the form of his demonstrations.

Little that is original can be expected in any elementary treatise on Descriptive Geometry. In Monge, Hachette or Vallée, I had ample materials for much more than I proposed giving, but I considered that my object would be best attained by taking as a guide an elementary treatise which had been found most practically useful in the country which gave birth to this branch of Geometry, and in which it has been most extensively cultivated and applied. Lacroix's elementary treatise, which was the first published, with the exception of Monge's Leçons, is still one of the best to the extent I proposed to go into the subject, and is well adapted for instruction; but on comparing it with Lefebure de Fourcy's I found that the latter was best calculated for the object I had in view. In the order of the Problems I have nearly followed De Fourcy, but I have endeavoured to present the principles of the science, and the construction of the Problems, in a form that should obviate the difficulties which may present themselves to a learner on his first entering on this subject. I have also added several problems not given in De Fourcy, the constructions of some of which are, as far as I know, original. The explanations which are here given of the principles of Descriptive Geometry and their application to most of the problems on Planes and Right lines will, I hope, enable the student to take up the subject, at the point at which it has been left, in De Fourcy or some of the more extensive works to which I have referred, and to pursue it to any extent that circumstances may render necessary.

In the Geometry of Planes I have likewise followed

nearly the order of the propositions in De Fourcy's introduction, but have by no means restricted myself to the form of the demonstrations there given. The demonstration which has heretofore been given of Prop. 47 (Simson's Euclid, xi. Prop. A; De Fourcy, Prop. 41)—an important proposition in the comparison of solids—is defective, since in it straight lines are assumed to meet, which in the case of one of the plane angles containing the trihedral angle being a right angle or obtuse, will not meet: the demonstration only applies to the case of the three plane angles being acute. I have here given a new demonstration in which this defect is obviated.

The Chapter on Horizontal Projection has been drawn up from M. Le Capitaine F. Noizet's "Mémoire sur la Géométrie appliquée au Dessin de la Fortification" (Mémorial de l'Officier du Génie, No. 6), a memoir that cannot be too strongly recommended to the attention of Military Engineers; to the various applications of the method, which are explained in the memoir, what is here given must be considered as merely introductory.

In explaining the principles of Isometric Perspective, I have had recourse to the late Professor Farish's original paper in the first volume of the Transactions of the Cambridge Philosophical Society.

Royal Military Academy,
January 1847.

CONTENTS.

PART III.

GEOMETRY.

Page

SUPPLEMENT TO EUCLID'S ELEMENTS OF PLANE GEOMETRY...... [1]

ELEMENTS OF THE GEOMETRY OF PLANES AND SOLIDS.

Definitions.. 1
Propositions. Planes and straight Lines 4
———— Dihedral angles 22
———— Solid angles 26
———— Trihedral angles 28
Theorems to be demonstrated 29*c*

ELEMENTARY PRINCIPLES OF DESCRIPTIVE GEOMETRY.

Introduction.. 31
Orthographic Projection 34
Principles of Descriptive Geometry...................... 40
Problems. Intersections of straight lines and Planes 43
———— Perpendicular straight lines and Planes 54
———— Angles of straight lines and Planes.............. 56
Miscellaneous Problems 71
Cases of the Trihedral angle 77
HORIZONTAL PROJECTION 82
Problems. Straight lines and Planes...................... 88
———— Surfaces 101

ISOMETRIC PERSPECTIVE 103

Geometry of Solids.

		Page
Propositions.	Equal solids	109
———	Parallelepipeds	111
———	Prisms	120
———	Pyramids	123
———	The Cylinder	128
———	The Cone	129
———	The Sphere	132

ERRATA.

Page 4, line 12, *for* Parallelopiped *read* Parallelepiped.
— 18, — 12, *for* (Prop. 30) *read* (Prop. 27).
— 18, — 18, *for* (v. 9) *read* (v. 11).
— 22, — 6 from bottom, *for* DC *read* DT.
— 24, — 7 from bottom, *for* Theor. *read* Prob.
— 28, — 5 from bottom, *for* ASP *read* ASB.

PART III.

GEOMETRY.

SUPPLEMENT

TO

EUCLID'S ELEMENTS OF PLANE GEOMETRY.

Definitions.

1. A *re-entering angle* of a figure is an angle having its angular point turned towards the interior of the figure.
2. The *Radius of a circle* is a straight line drawn from the centre to any point in the circumference.
3. An *Arc of a circle* is any portion of its circumference.
4. *The Chord* of an arc of a circle is the straight line joining the extremities of the arc; or the straight line which subtends the arc.
5. *A Tangent to an arc* at any point is a straight line touching the arc at that point.
6. The *Perimeter* of any figure is the whole length of the line, or lines, by which it is bounded.
7. The *Area* of any figure is the space contained within it.

Prop. I. Theor.

If from any magnitude there be taken away its half; from the remainder its half; and so on: there will at length remain a magnitude less than any magnitude of the same kind, however small.

Let AB (fig. 1) be any magnitude. If from AB there be taken away its half; from the remainder its half; and so on: there will at length remain a magnitude less than any magnitude C of the same kind, however small.

For C, however small, may be multiplied so as, at length, to be-

come greater than AB. Let DE then be a multiple of C, which is greater than AB, and let it contain the parts DF, FG, GH, HE, each equal to C. From AB take BI equal to its half; from the remainder AI, take KI equal to its half; and so on, until there be as many divisions in AB as there are in DE; and let the divisions in AB be BI, IK, KL, LA.

And because DE is greater than AB, and EH taken from DE is not greater than its half, but BI taken from AB is equal to its half, the remainder HD is greater than the remainder IA. Again, because HD is greater than IA, and HG is not greater than the half of HD, but IK is equal to the half of IA, the remainder GD is greater than the remainder KA. In like manner, because GD is greater than KA, and GF is not greater than the half of GD, but KL is the half of KA, the remainder FD is greater than the remainder LA. But FD is equal to C, therefore C is greater than LA, that is, LA is less than C. Wherefore, if from any magnitude, &c.: which was to be proved.

Prop. II. Theor.

If two points be joined by a straight line, and also by a series of straight lines making angles with each other, none of which angles are re-entering angles, then between this series of straight lines and the straight line joining the two points, other series of straight lines may be drawn, joining the points, which shall be shorter than the first series: and of such series of straight lines that nearer to the straight line joining the two points is always shorter than the more remote.

Let the points A, B (fig. 2) be joined by the straight line AB, and also by the series of straight lines AC, CD, DB, then between ACDB, and AB other series of straight lines may be drawn, joining A and B, which shall be shorter than ACDB.

In AC, CD take any points E, F, and join EF: then EF being less than EC and CF; AE, EF, FD, DB are together less than AC, CD, DB together; that is, AEFDB is shorter than ACDB. Again, in FD, DB take any points G, H, and join GH; then as before AEFGHB is shorter than AEFDB. For a like reason, taking in EF and GH, the points I, K, and joining IK, AEIKHB is shorter than AEFGHB. And so on, series of straight lines may be taken, each series nearer to B being shorter than any series more remote: which was to be proved.

Cor. 1. The straight line AB is shorter than any series of straight lines that can be drawn between A and B, on the same side of AB.

Cor. 2. From this it is evident that if a polygon be inscribed in another polygon, the perimeter of the inscribed polygon is less than the perimeter of the other.

Prop. III. Theor.

If the chord of an arc of a circle less than a semicircle be drawn, and from the extremities of the arc tangents be drawn meeting each other, the arc shall be greater than the chord, and less than the sum of the tangents between the point of their meeting and the points of contact.

Let ACB (fig. 3) be an arc of a circle, AB its chord, and AD, BD tangents to the circle at A and B, meeting each other in D: the arc ACB is greater than the chord AB, and less than the sum of the tangents AD, DB.

Bisect the arc ACB in C, draw the chords AC, CB, and at C draw the tangent ECF, meeting AD, BD in E and F; bisect the arcs AC, CB in G, H, draw the chords AG, GC, CH, HB, and at G and H, draw the tangents IGK and LHM; and so on, bisecting the arcs, and drawing their chords and tangents.

Then if the arc ACB be not greater than the chord AB, it must either be equal to AB or less than it; that is, AB will either be equal to the arc ACB or greater than it; and in either case the sum of AC and CB, being greater than AB, will be greater than the arc ACB. Again, AG, GC being greater than AC, and CH, HB greater than CB, the sum of the chords AG, GC, CH, HG will exceed the arc ACB more than the sum of the chords AC, CB exceeds that arc. In like manner, the sum of the chords AN, NG, GO, OC, CP, PH, HQ, QB will exceed the arc ACB, more than the sum of the chords AG, GC, CH, HG exceeds that arc; and so on: that is, the more nearly the chords of the arcs into which the arc ACB is divided, approach to coincidence with that arc, the more will their sum differ from it, which is manifestly absurd. Therefore the arc ACB is neither equal to the chord AB nor less than it: consequently the arc ACB is greater than the chord AB.

If the arc ACB be not less than the sum of the tangents AD, DB, it must be either equal to that sum or greater than it; and in either case, EF being less than ED, DF, the arc ACB will be greater than the sum of AE, EF, FB. Again, IK being less than IE, EK, and LM less than LF, FM, the arc ACB will exceed the sum of the tangents AI, IK, KL, LM, MB more than it exceeds the sum of the tangents AE, EF, FB. In like manner, the arc ACB will exceed the sum of the tangents AR, RS, ST, TU, UV, VW, WX, XY, YB more than it exceeds the sum of the tangents AI, IK, KL, LM, MB; and so on: that is, the more nearly the tangents of the arcs into which the arc ACB is divided approach to coincidence with that arc, the more will their sum differ from it, which again is manifestly absurd. Therefore the arc ACB is neither equal to the sum of AD, DB nor greater than it: consequently the arc ACB is less than the sum of the tangents AD, DB.

Wherefore, if the chord of any arc of a circle, &c.: which was to be demonstrated.

Cor. Hence the circumference of a circle is greater than the perimeter of any polygon inscribed in the circle, and less than the perimeter of any polygon described about the circle.

Prop. IV. Theor.

The perimeters of similar polygons inscribed in circles are to one another as the diameters of the circles; and their areas are to one another as the squares of the diameters.

Let ABCDEF, GHIKLM (fig. 4) be two similar polygons inscribed in the circles ABCDEF, GHIKLM, of which the diameters are AD, GK: the perimeter of the polygon ABCDEF is to the perimeter of the polygon GHIKLM as the diameter AD to the diameter GK.

Join BD, AC, HK, GI. Then because the polygon ABCDEF is similar to the polygon GHIKLM, the angle ABC is equal to the angle GHI, and AB is to BC as GH to HI (VI. Def. 1): therefore the two triangles ABC, GHI, having one angle in the one equal to one angle in the other, and the sides about the equal angles proportionals, are equiangular (VI. 6); and therefore the angle BCA is equal to the angle HIG: but BCA is equal to BDA, because they are in the same segment (III. 21); and, for the same reason, the angle HIG is equal to the angle HKG. Therefore the angle BDA is equal to HKG: and the right angle ABD (III. 31) is equal to the right angle GHK; wherefore the remaining angles in the triangles ABD, GHK are equal, and the triangles are equiangular to one another; therefore AB is to GH as AD to GK (VI. 4). And because AB is to GH as BC to HI, as CD to IK, as DE to KL, as EF to LM, and as FA to MG, therefore as one antecedent is to its consequent, so are all the antecedents taken together to all the consequents taken together (V. 12), that is, AB is to GH as the perimeter of the polygon ABCDEF to the perimeter of the polygon GHIKLM: but as AB is to GH so is AD to GK; consequently (V. 11) the perimeter of the polygon ABCDEF is to the perimeter of the polygon GHIKLM as the diameter AD to the diameter GK.

Also the area of the polygon ABCDEF is to the area of the polygon GHIKLM as the square of AD is to the square of GK.

Since AB is to GH as AD to GK, the duplicate ratio of AB to GH is the same as the duplicate ratio of AD to GK (V. Def. 10); but the ratio of the square of AD to the square of GK is the duplicate ratio of that which AD has to GK (VI. 20); and the ratio of the polygon ABCDEF to the polygon GHIKLM is the duplicate ratio of that which AB has to GH (VI. 20); therefore the polygon

ABCDEF is to the polygon GHIKLM as the square of AD to the square of GK (V. 11).

Wherefore the perimeters of similar polygons, &c.: which was to be proved.

Cor. Since equilateral polygons inscribed in a circle are equiangular (IV. 11, 15, 16), equilateral polygons, of the same number of sides, inscribed in circles, are similar; and, therefore, their perimeters are to each other as the diameters of the circles in which they are inscribed; and their areas are to each other as the square of the diameters.

Prop. V. Theor.

If, in a circle, an equilateral polygon be inscribed, and an equilateral polygon of the same number of sides be described about the circle, the perimeter of the inscribed polygon will be to the perimeter of the circumscribed, as the perpendicular from the centre on a side of the inscribed polygon is to the radius of the circle.

Let ABCDEFGH (fig. 5) be an equilateral polygon inscribed in the circle ACEG, of which the centre is R, and radius is RA: if through the points A, B, C, D, E, F, G, H straight lines IK, KL, LM, MN, NO, OP, PQ, QI be drawn, IKLMNOPQ will be an equilateral polygon of the same number of sides described about the circle (IV. 12, 15. 16. Cor.): the perimeter of the inscribed polygon ABCDEFGH will be to the perimeter of the circumscribed polygon IKLMNOPQ as the perpendicular from R upon AB is to the radius RA. Join RB, RK, RL.

Because KA is equal to KB (IV. 12), RA equal to RB and RK common to the two triangles KRA, KRB, the angle KRA is equal to the angle KRB; and the angle BRK is half the angle BRA. In like manner, the angle BRL is half the angle BRC: and since the angle ARB is equal to BRC, the angle BRK is equal to BRL. Since the two triangles TRA, TRB have the two sides AR, RT equal to the two BR, RT, each to each, and the angle ART equal to the angle BRT, the angle ATR is equal to the angle BTR; and RT is at right angles to AB. And because the triangles BRK, BRL have the two angles BRK, KBR equal to the two BRL, LBR, each to each, and BR common, RK is equal to RL. In like manner, it may be shown that RM, RN, RO, &c. are all equal; and therefore a circle described with the radius RK will circumscribe the polygon IKLMNOPQ. And since the equilateral polygons ABCDEFGH, IKLMNOPQ are inscribed in circles, their perimeters are to each other as the diameters of the circles in which they are described (Prop. 4), and consequently (V. 15) as the radii of those circles; therefore the perimeter of the polygon ABCDEFGH is to the perimeter of the polygon IKLMNOPQ as AR to RK, that is as RT to

RA, the triangles KRA and ART being similar. Wherefore, if in a circle, &c.: which was to be proved.

Cor. 1. The area of the inscribed polygon ABCD &c. is equal to the rectangle contained by half its perimeter and the perpendicular RT drawn from the centre R upon one of its sides. For the area of the triangle ABR is equal to the rectangle contained by BT and RT, that is by the half of AB and RT; and the same is true of all the other equal triangles which have their vertices in R, and which together make up the polygon ABCD &c.; therefore the whole polygon is equal to the rectangle contained by half the perimeter and the perpendicular RT. In the same manner it appears that the area of the circumscribing polygon IKLMN &c. is equal to the rectangle contained by half its perimeter and the radius RC.

Cor. 2. Since the equilateral polygons ABCD &c., IKLM &c., are similar polygons inscribed in circles, their areas are to each other as the squares of the diameters of their circumscribing circles (Prop. 4), and therefore as the squares of their radii; consequently the area of ABCD &c. is to the area of IKLM &c. as the square of RA to the square of RK, or as the square of RT to the square of RA.

Prop. VI. Theor.

In a given circle an equilateral polygon may be inscribed, and a similar polygon described about the circle, such that the perimeters of the polygons shall differ from one another by less than any given line, however small.

Let ABC (fig. 6) be a given circle, and DE a given line; an equilateral polygon may be inscribed in the circle ABC, and a similar polygon described about it, such that the difference of their perimeters shall be less than DE.

From DE cut off DF equal to the eighth part of DE (VI. 10), and in the diameter AC take CG equal to DF; through G draw IGH at right angles to AC; and from A apply the straight line AK equal to IH (IV. 1). Bisect the circumference ABC in B (III. 30), so that AB is a fourth part of the whole circumference of the circle; and from the circumference AB take away its half, and from the remainder its half, and so on, until the remaining circumference AL is found less than the circumference AK (Prop. 1). Bisect the circumference AL in M, and take the circumference AN equal to the circumference AM: join AL, and at A draw PQ touching the circle; find the centre O, and join OL, ON and OM cutting AL in R; and produce OM, ON to meet the tangent at A, in P and Q.

And because the arc AL was found by continued bisection, of the semi-circumference ABC, its half AB, and the half of that half,

and so on, AL will be contained a certain number of times exactly in the whole circumference: the straight line AL is therefore the side of an equilateral polygon inscribed in the circle ABC; and consequently (Prop. 5) PQ is the side of an equilateral polygon of the same number of sides described about ABC: the difference of the perimeters of the equilateral circumscribed and inscribed polygons of which PQ and AL are the sides, is less than the given line DE.

Let the perimeter of the equilateral circumscribed polygon of which PQ is the side be designated by C, and that of the similar inscribed polygon by I, then C is to I, as OM to OR (Prop. 5); and therefore by conversion (V. E), C is to its excess above I, as OM to MR, and therefore as eight times OM to eight times MR. And because the perimeter of a square described about the circle ABC is equal to four times the diameter AC, that is to eight times the radius OM, and the perimeter of the circumscribed equilateral polygon of eight sides is less than the perimeter of the circumscribed square (Prop. 2. Cor. 2); and again, the perimeter of the equilateral circumscribed polygon of sixteen sides is less than that of the polygon of eight sides; and so on; the perimeter of the equilateral polygon of which PQ is the side, that is C, is less than eight times OM: consequently (V. 16, A) the excess of C above I is less than eight times MR. And because AL is less than AK or HI, OR is greater than OG, and therefore MR is less than GC, that is than DF; and consequently eight times MR is less than eight times DF, that is than DE. And since it has been shown that the excess of C above I is less than eight times MR, with much greater reason is the excess of C above I less than DE.

Wherefore in a given circle, &c.: which was to be proved.

Cor. 1. Because the circumference of the circle is less than the perimeter of the circumscribing polygon, and greater than that of the inscribed polygon (Prop. 3. Cor.), it differs from either of these perimeters by less than they differ from one another, and therefore the difference between each of them and the circumference of the circle is less than DE. Consequently, however small a straight line may be, a polygon may be inscribed in the circle, and another described about it, the perimeter of each of which shall differ from the circumference of the circle by less than the given line.

Cor. 2. The straight line S, which is greater than the perimeter of any polygon that can be inscribed in the circle ABC, and less than the perimeter of any polygon that can be described about the circle, is equal to the circumference of the circle. For if not, let them be unequal; and first let S exceed the circumference, by the line Z. Then, because the perimeters of the circumscribing polygons are all greater than S, by hypothesis, and because S is greater than the circumference by Z, no polygon can be described about the circle but that its perimeter will exceed the circumference by a

line greater than Z, which by the preceding corollary is absurd. In the same manner, if S be less than the circumference by the line Z, it may be shown that no polygon can be inscribed in the circle but that its perimeter will be less than the circumference of the circle by a line greater than Z, which is likewise absurd. Therefore the straight line S and the circumference are not unequal, that is they are equal to one another.

PROP. VII. THEOR.

The circumferences of circles are to one another as their diameters.

Let ACE and GIL (fig. 4) be two circles of which the diameters are AD and GK; the circumference ACE is to the circumference GIL as AD to GK.

Let ABCDEF and GHIKLM be two equilateral polygons of the same number of sides inscribed in the circles ACE, GIL, and suppose P to be such a line that AD is to GK as the circumference ACE to P. Then, because the polygons ABCDEF and GHIKLM are equilateral and of the same number of sides they are similar, and their perimeters are as the diameters of the circles in which they are inscribed (Prop. 4). Therefore AD is to GK as the perimeter ABCDEF is to the perimeter GHIKLM; but AD is to GK as the circumference ABE is to P; and therefore the perimeter ABCDEF is to the perimeter GHIKLM as the circumference ABE is to P. And since the circumference ABE is greater than the perimeter ABCDEF (Prop. 3. Cor.), P is greater than the perimeter GHIKLM.

In like manner it may be demonstrated that P is less than the perimeter of any polygon described about the circle GIL; consequently P is equal to the circumference of the circle GIL (Prop. 6. Cor. 2). Now by hypothesis the perimeter ABE is to P as AD to GK; therefore the circumference ABE is to the circumference GIL as the diameter AD is to the diameter GK. Wherefore the circumferences of circles, &c.: which was to be proved.

PROP. VIII. THEOR.

In a given circle an equilateral polygon may be inscribed, and a similar polygon may be described about the circle such that the areas of the polygons shall differ from one another by less than any given area, however small.

Let ABC (fig. 7) be a given circle, of which L is the centre, and AC the diameter, and the square of D a given area; an equilateral polygon may be inscribed in the circle ABC, and a similar polygon described about it, such the difference of their areas shall be less than the square of D.

In the circle ABC apply the straight line AE equal to D; then by continued bisection of ABC, AB, &c., as in Proposition 6, find the arc AF less than AE; and as in that proposition, draw AF the side of the inscribed polygon, and IK the side of the circumscribed polygon: join FC. Then the area of the circumscribed polygon is to the area of the inscribed polygon as the square of LA to the square of LM (Prop. 5. Cor. 2), that is, by similar triangles, as the square of AC to the square of CF; or, calling the area of the circumscribed polygon P, and that of inscribed polygon Q, P is to Q as the square of AC to the square of CF; and therefore by conversion (V. E) P is to its excess above Q as the square of AC to its excess above the square of CF, that is, to the square of AF. But the square of AC, that is, the square described about the circle, is greater than the equilateral polygon of eight sides described about the circle, because it contains that polygon; and for the same reason, the polygon of eight sides is greater than the polygon of sixteen sides; and so on; therefore the square of AC is greater than any polygon described about the circle by the continual bisection of the arc AB: it is therefore greater than P. And since it has been shown that P is to its excess above Q, as the square of AC to the square of AF, and the square of AC is greater than P, the square of AF is greater than the excess of P above Q; with much greater reason is the square of AE, that is D, greater than the excess of P above Q: consequently the difference of the areas of the inscribed and circumscribed polygons is less than D, that is, than the given area. Wherefore in a given circle, &c.: which was to be proved.

Cor. 1. Because the area of the circle is less than that of the circumscribed and greater than that of the inscribed polygon, it differs from either of them by less than they differ from one another, and therefore the difference between each of the polygons and the circle is less than the given area, viz. the square of D. Consequently, however small any given area may be, a polygon may be inscribed in the circle, and another described about it, the area of each of which shall differ from the area of the circle by less than the given area.

Cor. 2. The area R, which is greater than the area of any polygon that can be inscribed in the circle ABC, and less than that of any polygon that can be described about it, is equal to the area of the circle. For if not, let them be unequal; and first let R exceed the circle by the area Z. Then because, by hypothesis, the areas of the circumscribing polygons are all greater than R, and R is greater than the circle by Z, no polygon can be described about the circle but that its area will exceed that of the circle by an area greater than Z, which, by the preceding corollary, is absurd. In the same manner, if R be less than the circle by Z, it is shown that no polygon can be inscribed in the circle but that its area will be

less than that of the circle by an area greater than Z, which is likewise absurd. Therefore the area R and the circle are not unequal, that is, they are equal to one another.

PROP. IX. THEOR.

The area of any circle is equal to the rectangle contained by the semi-diameter, and a straight line equal to half the circumference.

Let ABC (fig. 8) be a circle, of which the centre is L, and diameter AC; if in AC produced there be taken AS equal to half the circumference, the area of the circle is equal to the rectangle contained by LA and AS.

Let AF be the side of any equilateral polygon inscribed in the circle, and IK the side of an equilateral polygon of the same number of sides described about the circle, as in the last proposition. In AC produced take AD equal to half the perimeter of the polygon whose side is AF; and AE equal to half the perimeter of the polygon whose side is IK: then AL is greater than AD and less than AE (Prop. 2. Cor.). And since the area of the equilateral polygon whose side is IK is equal to the rectangle contained by LA and half its perimeter AE (Prop. 5. Cor. 1), the rectangle LA, AS, which is less than the rectangle LA, AE, is less than the area of any polygon that can be described about the circle ABC. Again, since the area of the polygon whose side is AF is equal to the rectangle contained by LM and half its perimeter AD, it is less than the rectangle LA, AD; with much greater reason then is it less than the rectangle LA, AS: the rectangle LA, AS is, therefore, greater than the area of any polygon inscribed in the circle ABC. But the same rectangle LA, AS has been shown to be less than the area of any polygon described about the circle; therefore the rectangle LA, AS is equal to the area of the circle ABC (Prop. 8. Cor. 2); that is, the area of the circle ABC is equal to the rectangle contained by its semi-diameter LA and half its circumference AS: which was to be proved.

PROP. X. THEOR.

The areas of circles are to one another in the duplicate ratio, or as the squares, of their diameters.

Let A and a be the areas of two circles of which the diameters are D and d; A will be to a in the duplicate ratio of D to d, or as the square of D to the square of d.

Let C and c be the circumferences of the two circles; then since by the last proposition A is equal to the rectangle contained by the half of D and the half of C, and a is equal to the rectangle contained by the half of d and the half of c, A will be to a as the rect-

angle contained by D and C is to the rectangle contained by d and c (V. 15). And since D is to d as C is to c (Prop. 7), the rectangle contained by D and C is similar to the rectangle contained by d and c; consequently the rectangle contained by D and C is to the rectangle contained by d and c in the duplicate ratio of D to d (VI. 20), or as the square of D to the square of d. Therefore A is to a in the duplicate ratio of D to d, or as the square of D to the square of d. Wherefore the areas of circles, &c.: which was to be proved.

Cor. 1. It follows from this demonstration also, that circles are to one another as the squares of their circumferences.

Cor. 2. If circles be described on the sides of a right-angled triangle ABC (fig. 9) as diameters, or as radii, the circle described on the side AC, subtending the right angle, is equal to the two circles described upon the other two sides AB, BC.

For the circle described on AB is to the circle described on AC, as the square of AB is to the square of AC; and the circle described on BC is to the circle described on AC, as the square of BC to the square of AC; therefore (V. 24) the circles described on AB and on BC, together, are to the circle described on AC, as the squares of AB and BC, together, are to the squares of AC. But the squares of AB and BC, together, are equal to the square of AC (I. 47); therefore (V. A) the circles described on AB and BC, together, are equal to the circle described on AC.

THE ELEMENTS

OF THE

GEOMETRY OF PLANES AND SOLIDS,

Definitions.

1. A solid is that which has length, breadth and thickness.
2. That which bounds a solid is a superficies.
3. *A straight line is perpendicular, or at right angles to a plane,* when it makes right angles with every straight line in that plane which meets it.

Reciprocally, a plane is said to be perpendicular, or at right angles to a straight line, when the straight line is perpendicular to the plane.

Thus the straight line AB (fig. 1), standing on the plane MN, and making right angles with every straight line, as CB, DB, EB, &c., in that plane, which meets it, is perpendicular or at right angles to the plane.

Reciprocally, the plane MN is perpendicular or at right angles to the line AB.

Straight lines which meet a plane and are not perpendicular to it are called *oblique.*

4. *The inclination of a straight line to a plane,* is the acute angle contained by that straight line, and another drawn from the point in which the first line meets the plane, to the point in which a perpendicular to the plane, drawn from any point of the first line above the plane, meets the same plane.

Thus, the inclination of the straight line AB (fig. 2) to the plane MN is the angle contained by AB and the straight line BD drawn

from the point B in which AB meets the plane, to the point D in which CD perpendicular to the plane, drawn from any point C in AB, meets the plane.

5. Straight lines drawn from the same point without a plane obliquely to the plane are said to be *equally distant* from the perpendicular drawn from that point to the plane, when the straight lines drawn from the intersections of the oblique lines with the plane, to the intersection of the perpendicular with the plane, are equal.

6. *A plane is perpendicular to a plane*, when the straight lines drawn in one of the planes perpendicular to the line of intersection of the two planes are perpendicular to the other plane.

Thus the plane AB (fig. 3) is perpendicular to the plane MN, when any straight lines DE, FG, HI drawn in the plane AB perpendicular to the line of intersection AC of the two planes are perpendicular to the plane MN.

7. *The inclination of a plane to a plane* is the acute angle contained by two straight lines drawn from any, the same point in the line of intersection of the planes, at right angles to it, one upon one plane, and the other upon the other plane.

Thus the inclination of the plane BA (fig. 4) to the plane MN is the acute angle EDF contained by the two straight lines DE, DF drawn from the same point D in the line of intersection CA of the two planes, at right angles to it, DE upon the plane MN, and DF upon the plane AB.

8. Two planes are said to have *the same* or *a like* inclination to one another which two other planes have, when the said angles of inclination are equal to one another.

9. *A straight line and a plane are parallel to each other* when, both being indefinitely produced, they do not meet.

10. *Parallel planes* are such as do not meet one another, though indefinitely produced.

11. *A Dihedral angle* is that which is contained by two planes which cut one another.

Thus the four angles which are contained by the two planes MN, AB (fig. 5), which cut one another, are Dihedral angles.

A dihedral angle is expressed by four letters, one in each of the planes and two in their intersection, these last two being placed between the two others: thus the dihedral angle formed by the planes MD, AD is expressed, "the angle MCDA," and the dihedral angle formed by the two planes MD, CB is expressed, "the angle MCDB."

The planes which form a dihedral angle are called its *faces*; and the intersection of these planes is called the *edge* of the dihedral angle: thus the planes MD, AD are the faces, and DC is the edge of the dihedral angle MDCA.

12. *A solid angle* is that which is made by the meeting of more than two plane angles, which are not in the same plane, in one point.

It is called a *Trihedral angle*, a *Tetrahedral angle*, a *Pentahedral angle*, &c., according as it is made by *three, four, five*, &c. plane angles.

Thus the angle which is made by the meeting in the point A (fig. 6) of the three plane angles BAC, BAD, CAD, not in the same plane, is a trihedral angle; the angle made by the meeting in the point A (fig. 7) of the four plane angles BAC, CAD, DAE, BAE, not in the same plane, is a tetrahedral angle; and both the former and the latter of these angles is a solid angle.

The plane angles and also the planes which form a solid angle are called its *faces*; the intersections of these planes are called the *edges*; and the point where these planes meet is called the *vertex* of the solid angle.

13. *Similar solid figures* are such as have all their solid angles equal, each to each, and are contained by the same number of similar plane rectilineal figures.

14. *A Pyramid* is a solid figure contained by plane triangles that are constituted between the sides of a plane rectilineal figure and a point above it in which the vertices of all these triangles meet.

15. *A Prism* is a solid figure contained by plane figures, of which two that are opposite are equal, similar, and parallel to one another; and the others are parallelograms.

16. *A Sphere* is a solid figure described by the revolution of a semicircle about its diameter, which remains fixed.

17. *The Axis of a sphere* is the fixed straight line about which the semicircle revolves.

18. *The Centre of a sphere* is the same with that of the semicircle.

19. *The Diameter of a sphere* is any straight line which passes through the centre, and is terminated both ways by the surface of the sphere.

20. *A Cone* is a solid figure contained by a circle and the surface described by the revolution of a straight line about the circumference, one extremity of the straight line being in a fixed point above the circle.

21. *The Axis* of a cone is the straight line joining the centre of the circle and the fixed point above it.

22. *The Base* of a cone is the circle about the circumference of which the straight line revolves.

23. *A Cylinder* is a solid figure described by the revolution of a right-angled parallelogram about one of its sides which remains fixed.

24. *The Axis of a cylinder* is the fixed straight line about which the parallelogram revolves.

25. *The Bases of a cylinder* are the circles described by the two revolving opposite sides of the parallelogram.

26. *Similar cylinders* are those which have their axes and the diameters of their bases proportionals.

27. *A Cube* is a solid figure contained by six equal squares.

28. *A Tetrahedron* is a solid figure contained by four equal and equilateral triangles.

29. *An Octahedron* is a solid figure contained by eight equal and equilateral triangles.

30. *A Dodecahedron* is a solid figure contained by twelve equal pentagons which are equilateral and equiangular.

31. *An Icosahedron* is a solid figure contained by twenty equal and equilateral triangles.

32. *A Parallelopiped* is a solid figure contained by six quadrilateral figures, whereof every opposite two are parallel.

Proposition I. Theorem.

One part of a straight line cannot be in a plane, and another part above it.

If it be possible, let AB (fig. 8), part of the straight line ABC, be in the plane MN, and the part BC above it. Then, since the straight line AB is in the plane MN, it can be produced in that plane; let it be produced to D; and let any plane pass through the straight line AD, and be turned about it until it pass through the point C. And because the points B, C are in this plane, the straight line BC is in it (I. Def. 7)*: therefore there are two straight lines, ABC, ABD in the same plane that have a common segment AB; which is impossible (I. 11. Cor.). Therefore one part of a straight line, &c.: which was to be proved.

Prop. II. Theor.

Two straight lines which cut one another are in one plane, and three straight lines which meet one another are in one plane.

Let two straight lines, AB, CD (fig. 9), cut one another in E; AB, CD shall be in one plane; and three straight lines EC, CB, BE, which meet one another shall be in one plane.

Let any plane pass through the straight line EB, and let the plane be turned about EB, produced if necessary, until it pass through the point C. Then because the points E, C are in this plane the straight line EC is in it (I. Def. 7): for the same reason, the straight line BC is in the same: and by the hypothesis, EB is in it: therefore the three straight lines EC, CB, BE are in one plane: but in the plane in which EC, EB are, in the same are CD,

* This and similar references are to the Book and Definition or Proposition in Euclid's 'Elements.'

AB (Prop. 1)*: therefore AB, CD are in one plane: which was to be proved.

Prop. III. Theor.

If two planes cut one another, their common section is a straight line.

Let two planes, AB, BC (fig. 10), cut one another, and let the line DB be their common section: DB shall be a straight line.

If it be not, from the point D to B draw, in the plane AB, the straight line DEB (I. Post. 1), and in the plane BC, the straight line DFB. Then two straight lines DEB, DFB have the same extremities and therefore include a space betwixt them: which is impossible (I. Ax. 10): therefore BD the common section of the planes AB, BC, cannot but be a straight line: which was to be proved.

Prop. IV. Theor.

If a straight line stand at right angles to each of two straight lines in the point of their intersection, it shall also be at right angles to the plane which passes through them, that is, to the plane in which they are.

Let the straight line EF (fig. 11) stand at right angles to each of the straight lines AB, CD, in E, the point of their intersection: EF shall also be at right angles to the plane MN passing through AB, CD.

Take the straight lines AE, EB, CE, ED, all equal to one another; join AD, CB; and through E draw, in the plane in which are AB, CD, any straight line GEH. Then from any point F, in EF, draw FA, FG, FD, FC, FH, FB.

Because the two straight lines AE, ED are equal to the two BE, EC, each to each, and that they contain equal angles AED, BEC (I. 15), the base AD is equal to the base BC (I. 4), and the angle DAE to the angle EBC: and the angle AEG is equal to the angle BEH (I. 15): therefore the triangles AEG, BEH have two angles of the one equal to two angles of the other, each to each, and the sides AE, EB, adjacent to the equal angles, equal to one another: wherefore they have their other sides equal (I. 26): therefore GE is equal to EH, and AG to BH. And because AE is equal to EB, and FE common and at right angles to them, the base AF is equal to the base FB (I. 4); for the same reason, CF is equal to FD. And because AD is equal to BC, and AF to FB, the two sides FA, AD are equal to the two FB, BC, each to each; and the base DF was proved equal to the base FC; therefore the angle FAD is equal to the angle FBC (I. 8). Again, it was proved that GA is equal

* This and similar references are to Propositions in these Elements.

to BH, and also AF to FB; therefore FA and AG are equal to FB and BH, each to each; and the angle FAG has been proved equal to the angle FBH; therefore the base GF is equal to the base FH (I. 4): but it was proved that GE is equal to EH, and EF is common; therefore GE, EF are equal to HE, EF, each to each; and the base GF is equal to the base FH; therefore the angle GEF is equal to the angle HEF (I. 8); and consequently each of these angles is a right angle (I. Def. 10). Therefore FE makes right angles with GH, that is, with any straight line drawn through E in the plane passing through AB, CD. In like manner, it may be proved, that FE makes right angles with every straight line which meets it in that plane. But a straight line is at right angles to a plane when it makes right angles with every straight line which meets it in that plane (Def. 3): therefore EF is at right angles to the plane in which are AB, CD: which was to be proved.

Prop. V. Prob.

From a given point without a given plane to draw a straight line perpendicular to the plane.

Let A (fig. 12) be the given point without the plane MN: it is required to draw from the point A a straight line perpendicular to the plane MN.

In the plane MN take any point B, and join AB; then if AB is perpendicular to the plane, what is required is now done. But if not, a straight line BC may be drawn in the plane MN not at right angles to AB (Def. 3). From A, in the plane ABC, draw AD perpendicular to BC (I. 12); and from D, draw, in the plane MN, DE perpendicular to BD (I. 11). Then if AD be at right angles to DE, it is at right angles to the plane EDB or MN (Prop. 4), and is the perpendicular required. But if not, from A, draw, in the plane ADE, AF perpendicular to DE (I. 12): AF is the perpendicular required.

Join FB. And because the angle ADB is a right angle, the square of AB is equal to the squares of AD and DB (I. 47); and because the angle AFD is a right angle, the square of AD is equal to the squares of AF and FD (I. 47): therefore the square of AB is equal to the squares of AF, FD and DB. But the square of FB is equal to the squares of FD and DB, because the angle FDB is a right angle: consequently the square of AB is equal to the squares of AF and FB; and therefore the angle AFB is a right angle (I. 48). And because AF is at right angles to DF and BF at their point of intersection, it is at right angles to the plane BFD (Prop. 4), that is, to the plane MN; and it is drawn from the given point A: which was required to be done.

Prop. VI. Theor.

From the same point without a plane, there can be only one perpendicular to the plane.

For, if possible, let the two straight lines AB and AC (fig. 13), meeting the plane MN in the point B and C, be both perpendicular to the plane; and join BC.

Because AB is perpendicular to the plane MN, and CB in that plane meets it, the angle ABC is a right angle (Def. 3); and because AC is perpendicular to the plane MN, and BC in that plane meets it, the angle ACB is a right angle. Consequently the two angles ABC, ACB of the triangle BAC are two right angles, which is impossible (I. 17). Therefore from the same point, &c.: which was to be proved.

Prop. VII. Prob.

From a given point in a given plane to draw a straight line perpendicular to the plane.

Let A (fig. 14) be the given point in the given plane MN: it is required to draw from the point A a straight line perpendicular to the plane MN.

From any point B without the plane MN draw BC perpendicular to it (Prop. 5) and meeting it in C; and join AC.

At A in the plane BCA draw AD at right angles to CA (I. 11): AD is perpendicular to the plane MN.

At C in the plane MN draw CE at right angles to AC; and join AE. Draw CD, in the plane BCA, to meet AD: and join DE.

Because BC is perpendicular to the plane MN, the angle BCE is a right angle (Def. 3): and because EC is at right angles to CB and CA, it is at right angles to the plane BCA (Prop. 4), and therefore to the straight line CD which meets it in that plane (Def. 3). And because the angle DCE is a right angle, the square of DE is equal to the squares of DC and CE (I. 47): but the square of DC is equal to the squares of DA and AC, because the angle DAC is a right angle; therefore the square of DE is equal to the squares of DA, AC and CE: and the square of AE is equal to the squares of AC and CE, because ACE is a right angle: consequently the square of DE is equal to the squares of DA and AE; and therefore the angle DAE is a right angle (I. 48). And because DA is at right angles to the two straight lines CA, EA at their point of intersection A, it is at right angles to the plane MN in which they are (Prop. 4); and it is drawn from the given point A; which was required to be done.

Prop. VIII. Theor.

From the same point in a plane, there can be only one perpendicular to the plane, upon the same side of it.

For if it be possible, let the two straight lines AB, AC (fig. 15) be perpendicular to the plane MN, from the same point A, and upon the same side of it; and let the common section of the plane passing through BA and CA be the straight line DAE (Prop. 3); so that BA, CA and DAE are in one plane. And because BA is perpendicular to the plane MN, and EA meets it in that plane, the angle BAE is a right angle (Def. 3). For the same reason the angle CAE is a right angle. Wherefore the angle BAE is equal to the angle CAE, in the same plane; the greater to the less, which is impossible. Therefore from the same point, &c.: which was to be proved.

Prop. IX. Theor.

The perpendicular let fall from a given point to a given plane is the shortest straight line that can be drawn from the point to the plane.

Let A (fig. 16) be the given point, MN the given plane, and AB the perpendicular from A to the plane MN: AB is the shortest line that can be drawn from A to the plane MN.

For from A to the plane MN draw any other line AC, and join CB. Then since AB is perpendicular to the plane MN, the angle ABC is a right angle (Def. 3); and therefore the angle ACB is less than a right angle (I. 17). Consequently AB is less than AC (I. 19): which was to be proved.

Scholium. The perpendicular AB measures the distance of the point A from the plane MN.

Prop. X. Prob.

Through a given point to draw a plane perpendicular to a given straight line.

Let AB be the given straight line, and O the given point (figs. 17, 18). First, let the point O be in the straight line AB (fig. 17).

In two planes AOC, AOD passing through AB, draw OC, OD at right angles to AB (I. 11). Then AO being at right angles to each of the straight lines OC, OD at their point of intersection O, it is perpendicular to the plane MON passing through these lines (Prop. 4). Consequently the plane MON is perpendicular to the straight line AB (Def. 3); and it is drawn through the given point O: which was required to be done.

Secondly, let the point O be without the straight line AB (fig. 18).

From the point O, draw OC at right angles to AB (I. 12); and from the point C, in a plane passing through AB, draw CD also at

right angles to AB (I. 11). Then AB being at right angles to each of the lines CO, CD at their point of meeting C, it is perpendicular to the plane MON passing through these lines (Prop. 4). Consequently the plane MON is perpendicular to the straight line AB (Def. 3); and it is drawn through the given point O: which was required to be done.

PROP. XI. THEOR.

Through a given point, only one plane can be drawn perpendicular to a given straight line.

First, let the given point O be in the given straight line AB (fig. 17): then besides the plane MON, drawn as in the last proposition, no other plane can be drawn through O, perpendicular to AB.

For, if possible, let another plane as MOR be perpendicular to AB, then a plane passing through AB may be drawn, intersecting the planes MN, MR in two straight lines as OE and OF; and consequently each of the angles AOE, AOF, in the same plane, is a right angle (Def. 3); and they are equal to each other (I. Ax. 11), which is impossible (I. Ax. 9).

Secondly, let the point O be without the given straight line AB (fig. 18): then besides the plane MON, drawn as in the last proposition, no other plane can be drawn through O perpendicular to AB.

For if another plane as MOR were perpendicular to AB, then a plane passing through AB may be drawn intersecting the planes MN, MR in two straight lines as OC and OE; and consequently each of the angles ACO, AEO would be a right angle (Def. 3), which is impossible (I. 16).

Therefore through a given point, &c.: which was to be proved.

PROP. XII. THEOR.

If three straight lines meet all in one point, and a straight line stand at right angles to each of them in that point; these three straight lines are in one and the same plane.

Let the straight line AB (fig. 19) stand at right angles to each of the straight lines BC, BD, BE, in B the point where they meet: BC, BD, BE are in one and the same plane.

If not, let, if it be possible, BD and BE be in one plane, and BC be above it; and let a plane pass through AB, BC, the common section of which, with the plane in which BD and BE are, is a straight (Prop. 3) line; let this be BF: therefore the three straight lines, AB, BC, BF, are all in one plane, viz. that which passes through AB, BC: and because AB stands at right angles to each of the straight lines BD, BE, it is also at right angles (Prop. 4) to the plane passing through them; and therefore makes right angles with every straight line, in that plane, which meets it (Def. 3): but

BF, which is in that plane, meets it; therefore the angle ABF is a right angle: but the angle ABC, by the hypothesis, is also a right angle; therefore the angle ABF is equal to the angle ABC, and they are both in the same plane, which is (I. Ax. 9) impossible; therefore the straight line BC is not above the plane in which are BD and BE: wherefore the three straight lines BC, BD, BE are in one and the same plane: which was to be proved.

Cor. From this it follows, that all straight lines which are perpendicular to a given straight line, at a given point, are in the plane which is perpendicular to the given line, at that point.

Let the straight lines BC, BD, BE, BG, &c. be straight lines perpendicular to the straight line AB, at the point B: BC, BD, BE, BG, &c. are in the plane which is perpendicular to AB, at the point B.

For the line AB being at right angles to each of the lines BC, BD, BE, at their point of meeting, BC, BD, BE, are by this proposition in the same plane. Similarly BD, BE, BG are in the same plane. Consequently all the lines BC, BD, BE, BG are in the same plane. And since AB is perpendicular to the plane in which are BC, BD, BE, BG, &c. (Prop. 4), this plane is perpendicular to AB (Def. 3).

Prop. XIII. Theor.

If a plane be perpendicular to a straight line at its middle point, every point in the plane will be equally distant from the extremities of the line.

Let the plane MN (fig. 20) be perpendicular to the straight line AB, at its middle point C; any point H in the plane MN is equally distant from A and B.

For join HC, HA, HB. Then because BC is equal to CA, and CH common to the two triangles BCH, ACH; and the angles BCH, ACH are right angles (Def. 3), BH is equal AH: which was to be proved.

Cor. Any point I, without the plane MN, is unequally distant from B and A. For BI is less than BH and HI (I. 20), that is, than AH and HI, or than AI.

Prop. XIV. Theor.

Of straight lines drawn from a point without a plane, oblique to the plane, those are equal which are equally distant from the perpendicular drawn from the same point to the plane; and that which is nearer to the perpendicular is less than the one more remote.

Also, conversely, equal straight lines drawn from a point without a plane, oblique to the plane, are equally distant from the perpendicular drawn from the same point to the plane; and of unequal straight lines drawn from the same point to the plane, the less is nearer to the perpendicular than the greater.

Let AC, AD, AE (fig. 21) be straight lines drawn from the point A oblique to the plane MN, of which AC, AD are equally distant from AB, perpendicular to the plane MN, that is, CB being joined and also DB, CB is equal to DB: AC shall be equal to AD.

Because CB is equal to DB, and AB is common to the two triangles CBA, DBA, and the angles CBA, DBA are right angles, AC is equal AD (I. 4).

Next let AC be nearer to AB than AE is, that is, joining EB, let CB be less than EB, AC is less than AE.

For the square of BC is less than the square of BE; therefore the squares of CB and BA are less than the squares of EB and BA; that is (I. 47), the square of CA is less than the square of EA, consequently CA is less than EA.

Conversely, if AC is equal to AD, BC is equal to BD.

For since the square of AC is equal to the square of AD, the squares of AB, BC are equal to the squares of AB, BD (I. 47). Therefore the square of BC is equal to the square of BD; and consequently BC is equal to BD.

Also, if AC is less than AE, BC is less than BE.

For the square of AC being less than the square of AE, the squares of AB, BC are less than the squares of AB, BE (I. 47). Therefore the square of BC is less than the square of BE; and consequently BC less than BE.

Therefore of straight lines drawn, &c.: which was to be proved.

PROP. XV. THEOR.

If from any point in a straight line oblique to a plane, a perpendicular be drawn to the plane, and the intersections of the oblique line and the perpendicular with the plane be joined by a straight line, the oblique straight line will form a less angle with the line joining the intersections than with any other line meeting it in the plane.

From the point A (fig. 22), in the straight line AB which is oblique to the plane MN, let AC be drawn perpendicular to the plane, and BC be joined; the angle which AB makes with BC shall be less than that which it makes with any other line BD meeting it in the plane MN.

For take BD equal to BC, and join AD.

Then because AC is perpendicular, and AD oblique to the plane, AC is less than AD (Prop. 9). And because BC is equal to BD and AB common to the two triangles ABC, ABD, and AC is less than AD, the angle ABC is less than the angle ABD (I. 25): which was to be proved.

Scholium. It is for this reason that the angle ABC is considered to measure the inclination of the straight line AB to the plane MN.

PROP. XVI. THEOR.

If from the foot of the perpendicular to a given plane a perpendicular be let fall upon any straight line in the plane, a straight line drawn from any point in the perpendicular to the plane, to the foot of the perpendicular on the straight line, will be perpendicular to that line; and the plane passing through the perpendicular to the given plane and that on the straight line, will be perpendicular to that line.

Let AB (fig. 23) be perpendicular to the plane MN, and from B let BE be drawn perpendicular to the straight line CD in the plane; let AE be joined: AE is perpendicular to CD.

Take EC and ED equal to each other, and join BC, BD, AC, AD.

Then because CE is equal to DE, and BE is common to the two triangles BEC, BED, and at right angles to DC, BC is equal to BD (I. 4). And because the oblique lines AC and AD are equally distant from the perpendicular AB (Def. 5), AC is equal to AD (Prop. 14). And because CE is equal to DE, AE common to the two triangles AEC, AED, and AC equal to AD, the angle AEC is equal to the angle AED (I. 8); and therefore AE is at right angles to DC (I. Def. 10).

Also the plane ABE is perpendicular to DC.

For DC is at right angles to BE and AE, and therefore it is perpendicular to the plane ABE (Prop. 4).

Wherefore, if from the foot of the perpendicular, &c.: which was to be proved.

PROP. XVII. THEOR.

If two straight lines, be at right angles to the same plane they shall be parallel to one another.

Let the straight lines AB, CD (fig. 24), meeting the plane MN in the points B and D, be at right angles to it: AB shall be parallel to CD.

Join BD; draw in the plane MN, DE at right angles to BD; and join AD. Then because from B, the foot of the perpendicular to the plane MN, BD is drawn perpendicular to DE in the plane, and AD is drawn, from a point in AB, to D, AD is perpendicular to DE (Prop. 16); and ED is at right angles to AD. But it is also at right angles to each of the two BD, CD at the point D in which the three straight lines BD, AD, CD meet; therefore these three straight lines are all in the same plane (Prop. 12). But AB is in the plane in which are BD, AD (Prop. 2); therefore AB, BD, CD are in one plane: and since each of the angles ABD, CDB is a right angle, AB is parallel to CD (I. 28). Wherefore, if two straight lines, &c.: which was to be proved.

Prop. XVIII. Theor.

If two straight lines be parallel, and one of them be at right angles to a plane, the other also shall be at right angles to the same plane.

Let AB, CD (fig. 24) be two parallel straight lines meeting the plane MN in B, D, and let one of them, AB, be at right angles to the plane; the other, CD, shall be at right angles to the same plane.

Join BD, AD; then AB and CD being parallel, they are in the same plane (I. Def. 35); and BD and DA are also in that plane (I. Def. 7). In the plane MN draw DE at right angles to BD.

Then because AB is perpendicular to the plane MN, DE is perpendicular to the plane ABD (Prop. 16), that is, to the plane ABDC; consequently CDE is a right angle (Def. 3). And because BD meets the parallels AB, CD, the angles ABD, CDB are together equal to two right angles (I. 29); but ABD is a right angle (Def. 3); therefore CDB is also a right angle. And it has been shown that the angle CDE is a right angle; therefore CD is at right angles to BD, DE at their point of intersection, and is consequently at right angles to the plane BDE, that is, to the plane MN. Therefore, if two straight lines, &c.: which was to be proved.

Prop. XIX. Theor.

Two straight lines, which are each of them parallel to the same straight line, and not both in one plane with it, are parallel to one another.

Let AB, CD (fig. 25) be each of them parallel to EF, and not both in one plane with it: AB shall be parallel to CD.

In EF take any point G, from which draw, in the plane passing through EF, AB, the straight line GH at right angles to EF (I. 11); and in the plane passing through EF, CD, draw GK at right angles to the same EF.

Because EF is perpendicular both to GH and GK, EF is perpendicular to the plane HGK passing through them (Prop. 4): and EF is parallel to AB; therefore AB is at right angles to the plane HGK (Prop. 18). For the same reason CD is likewise at right angles to the plane HGK. Therefore AB, CD are each of them at right angles to the plane HGK. But if two straight lines are at right angles to the same plane, they are parallel to one another (Prop. 17): therefore AB is parallel to CD. Wherefore two straight lines, &c.; which was to be proved.

Cor. If through two parallel straight lines AB, CD, planes which cut each other be drawn, their intersection EF will be parallel to AB and CD. For from any point E, in EF, draw, in the plane ABFE, a straight line parallel to AB; this line will, by the proposition, be parallel to CD, and therefore in the plane DCEF: it must consequently be the intersection of the two planes passing through AB, CD.

Prop. XX. Theor.

If two straight lines meeting one another be parallel to two others that meet one another, and are not in the same plane with the first two, the first two and the other two shall contain equal angles.

Let the two straight lines AB, BC (fig. 26.), which meet one another, be parallel to the two straight lines DE, EF that meet one another, and are not in the same plane with AB, BC. The angle ABC shall be equal to the angle DEF.

Take BA, BC, ED, EF all equal to one another; and join AD, CF, BE, AC, DF.

Then because BA is equal and parallel to ED, therefore AD is both equal and parallel to BE (I. 33). For the same reason, CF is equal and parallel to BE. Therefore AD and CF are each of them equal and parallel to BE; and they are not both in one plane with it, therefore AD is parallel to CF (Prop. 19); and it is also equal to it (I. Ax. 1); and AC, DF join AD, CF towards the same parts; therefore AC is equal and parallel to DF (I. 33). And because AB, BC are equal to DE, EF, each to each, and the base AC to the base DF; the angle ABC is equal to the angle DEF (I. 8). Therefore, if two straight lines, &c.: which was to be proved.

Prop. XXI. Theor.

If a straight line without a plane be parallel to a straight line in the plane, it will be parallel to the plane.

Let the straight line CD (fig. 27) without the plane MN be parallel to the straight line AB in that plane; CD will be parallel to the plane MN. For CD being in the same plane as AB (I. Def. 35), if, being produced, it meet the plane MN, it must meet it in the line AB produced; and this is impossible, since CD is parallel to AB. Therefore CD is parallel to the plane MN: which was to be proved.

Prop. XXII. Theor.

If a straight line be perpendicular to a plane, any straight line perpendicular to this line will be parallel to the plane.

Let the straight line AB be perpendicular to the plane MN, and the straight line AC be perpendicular to AB; AC is parallel to the plane MN.

Through AC and AB (fig. 28), let the plane AD be drawn, intersecting the plane MN in BD. Then since AB is perpendicular to the plane MN, ABD is a right angle; and BAC is a right angle by hypothesis: therefore AC is parallel to BD (I. 28), and consequently to the plane MN (Prop. 21): which was to be proved.

Prop. XXIII. Theor.

If a straight line be parallel to a plane, the intersection with this plane of any plane passing through the line will be parallel to the line.

Let the straight line CD (fig. 27) be parallel to the plane MN, and let AB be the intersection of a plane passing through CD with the plane MN: AB is parallel to CD.

Since CD is parallel to the plane it cannot meet any line AB in the plane. AB being in the same plane with CD is therefore parallel to it (I. Def. 35): which was to be proved.

Cor. 1. CD being parallel to the plane MN, if through any point A, in the plane MN, a straight line AB be drawn parallel to CD, it will be wholly in the plane MN. For this line must be the intersection of the plane passing through CD and A, with the plane MN.

Cor. 2. A straight line parallel to each of two planes which cut one another is parallel to their intersection. For if through any point of the intersection of the two planes a straight line be drawn parallel to the former line, it will be in each of the two planes, and will therefore be their intersection.

Prop. XXIV. Theor.

Parallel straight lines, contained between a plane and a straight line parallel to it, are equal.

Let AC and BD (fig. 27) be parallel straight lines, contained between the plane MN and the straight line CD parallel to MN; AC and BD are equal.

For as the plane of the parallels AC, BD intersects the plane MN in a line AB parallel to CD (Prop. 23), ABDC is a parallelogram: consequently AC is equal to BD (I. 34): which was to be proved.

Prop. XXV. Theor.

Planes, to which the same straight line is perpendicular, are parallel to one another.

Let the straight line AB (fig. 29) be perpendicular to each of the planes CD, EF: these planes shall be parallel to one another.

If not, they shall meet one another when produced: let them meet; their common section is a straight line GH, in which take any point K, and join AK, BK. Then, because AB is perpendicular to the plane EF, it is perpendicular to the straight line BK which is in that plane (Def. 3): therefore ABK is a right angle. For the same reason BAK is a right angle: wherefore the two angles ABK, BAK of the triangle ABK are equal to two right angles, which is impossible (I. 17): therefore the planes CD, EF, though produced, do not meet one another; that is (Def. 10), they are parallel. Therefore planes, &c.: which was to be proved.

Prop. XXVI. Theor.

If two straight lines meeting one another be parallel to two other straight lines which meet one another, but are not in the same plane with the first two; the plane which passes through these is parallel to the plane passing through the others.

Let AB, BC (fig. 30), two straight lines meeting one another, be parallel to DE, EF that meet one another, but are not in the same plane with AB, BC: the plane passing through AB, BC shall be parallel to that through DE, EF.

From the point B draw BG perpendicular to the plane which passes through DE, EF (Prop. 5), and let it meet that plane in G; and through G draw GH parallel to ED, and GK parallel to EF (I. 31).

And because BG is perpendicular to the plane through DE, EF, it makes right angles with every straight line meeting it in that plane (Def. 3): but the straight lines GH, GK in that plane meet it; therefore each of the angles BGH, BGK is a right angle: and because BA is parallel to GH (Prop. 19) (for each of them is parallel to DE, and they are not both in the same plane with it), the angles GBA, BGH are together equal to two right angles (I. 29): and BGH is a right angle; therefore also GBA is a right angle, and GB is perpendicular to BA. For the same reason, GB is perpendicular to BC. Since therefore the straight line GB stands at right angles to the two straight lines BA, BC that cut one another in B, GB is perpendicular to the plane through BA, BC (Prop. 4): and, by construction, it is perpendicular to the plane through DE, EF; therefore BG is perpendicular to each of the planes through AB, BC, and DE, EF: but planes to which the same straight line is perpendicular, are parallel to one another (Prop. 25); therefore the plane through AB, BC is parallel to the plane through DE, EF: which was to be proved.

Cor. Through two straight lines AB, EF, not in the same plane, two planes parallel to each other may always be drawn. For taking any point B in AB, and any point E in EF, draw BC parallel to EF, and ED parallel to AB, the plane passing through AB, BC, and that through DE, EF will be parallel.

Prop. XXVII. Theor.

If two parallel planes be cut by another plane, their common sections with it are parallels.

Let the parallel planes AB, CD (fig. 31) be cut by the plane EFHG, and let their common sections with it be EF, GH: EF shall be parallel to GH.

For if it be not, EF, GH will meet, if produced, either on the side

of FH, or EG. First, let them be produced on the side of FH, and meet in the point K: therefore, since EFK is in the plane AB, every point in EFK is in that plane (Prop. 1): and K is a point in EFK; therefore K is in the plane AB: for the same reason K is also in the plane CD: wherefore the planes AB, CD produced meet one another: but they do not meet, since they are parallel by the hypothesis; therefore the straight lines EF, GH do not meet when produced on the side of FH. In the same manner it may be proved that EF, GH do not meet when produced on the side of EG. But straight lines which are in the same plane, and do not meet, though produced either way, are parallel; therefore EF is parallel to GH: which was to be proved.

Prop. XXVIII. Theor.

A straight line which is perpendicular to one of two parallel planes is perpendicular to the other.

Let the straight line AB (fig. 32) be perpendicular to MN, one of the two parallel planes MN, PQ; AB is also perpendicular to the other plane PQ.

From the point B, where the straight line AB meets the plane PQ, draw any straight line BC; then the intersection AD of the plane passing through AB and BC will be parallel to BC (Prop. 26). And since DAB is a right angle (Def. 3), ABC is also a right angle (I. 28). In the same manner it may be shown that AB is at right angles to any other straight line meeting it in the plane PQ: it is therefore (Def. 3) perpendicular to the plane PQ: which was to be proved.

Prop. XXIX. Theor.

Planes which are parallel to the same plane are parallel to each other.

Let each of the planes P, Q (fig. 33) be parallel to the plane R; the planes P and Q are parallel to each other.

From any point C, in the plane R, draw the straight line CBA perpendicular to R (Prop. 7). Then CBA is perpendicular to each of the planes P and Q (Prop. 28); and, consequently, the planes P and Q are parallel (Prop. 25): which was to be proved.

Prop. XXX. Theor.

Parallel straight lines contained between parallel planes are equal.

Let AB, CD (fig. 34) be parallel straight lines contained between the parallel planes MN, PQ: AB and CD are equal.

For AB and CD being parallel, they are in the same plane (I. Def. 35); and the intersections AC, BD of this plane with the parallel planes MN, PQ are parallel (Prop. 27): consequently ABDC

is a parallelogram, and therefore AB and CD are equal: which was to be proved.

Prop. XXXI. Theor.

If two straight lines be cut by parallel planes, they shall be cut in the same ratio.

Let the straight lines AB, CD (fig. 35) be cut by the parallel planes GH, KL, MN, in the points A, E, B; C, F, D: as AE is to EB, so shall CF be to FD.

Join AC, BD, AD, and let AD meet the plane KL in the point X; and join EX, XF. Because the two parallel planes KL, MN are cut by the plane EBDX, the common sections EX, BD are parallel (Prop. 30): for the same reason, because the two parallel planes GH, KL are cut by the plane AXFC, the common sections AC, XF are parallel. And because EX is parallel to BD, a side of the triangle ABD; as AE to EB, so is AX to XD (VI. 2): again, because XF is parallel to AC, a side of the triangle ADC; as AX to XD, so is CF to FD: and it was proved that AX is to XD, as AE to EB; therefore, as AE to EB, so is CF to FD (v. 9). Wherefore, if two straight lines, &c.: which was to be proved.

Prop. XXXII. Theor.

If a straight line be at right angles to a plane, every plane which passes through it shall be at right angles to that plane.

Let the straight line AB (fig. 36) be at right angles to the plane MN: every plane which passes through AB shall be at right angles to the plane MN.

Let any plane DE pass through AB, and let CE be the common section of the planes DE, MN; take any point F in CE, from which draw FG in the plane DE at right angles to CE (I. 11).

Because AB is perpendicular to the plane MN, therefore it is also perpendicular to every straight line in that plane, meeting it (Def. 3): and consequently it is perpendicular to CE: wherefore ABF is a right angle: but GFB by construction is likewise a right angle; therefore AB is parallel to FG (I. 28): and AB is at right angles to the plane MN; therefore FG is also at right angles to the same plane (Prop. 18). But one plane is at right angles to another plane when the straight lines drawn in one of the planes, at right angles to their common section, are also at right angles (Def. 6) to the other plane; and any straight line FG in the plane DE, which is at right angles to CE, the common section of the planes, has been proved to be perpendicular to the other plane MN; therefore the plane DE is at right angles to the plane MN. In like manner, it may be proved that all planes which pass through AB are at right angles to the plane MN: which was to be proved.

Cor. 1. The plane CH being perpendicular to the plane MN, the latter will be perpendicular to the former.

For AB being perpendicular to CE (Def. 3), in the plane MN draw BK at right angles to CE: then AB is at right angles to BK; therefore KB is at right angles to BA and BC, and consequently it is perpendicular to the plane ABC (Prop. 4); and being perpendicular to the common section CE, the plane MN is perpendicular to the plane CH (Def. 6).

Cor. 2. If at a point, three straight lines are severally perpendicular, each to the other two, then each being perpendicular to the plane of the other two, the three planes are perpendicular to each other.

Prop. XXXIII. Theor.

When two planes are perpendicular to each other, every perpendicular to one of the planes drawn from a point in the other is contained wholly in the latter.

Let the planes MN, CH (fig. 37) be perpendicular to each other, a straight line drawn perpendicular to the plane MN from any point A, in the plane CH will be contained wholly in the plane CH.

For, from A, draw AB perpendicular to CE, the intersection of the two planes: then AB is perpendicular to the plane MN (Def. 6); and besides AB, no other straight line can be drawn from the point A perpendicular to the plane MN (Prop. 6); and AB is wholly in the plane CH: which was to be demonstrated.

Cor. Through a straight line not perpendicular to a plane only one plane can be drawn perpendicular to that plane. For the perpendicular plane must contain the given straight line, and likewise a straight line drawn from a point in it perpendicular to the given plane: and two straight lines which intersect, determine a plane (Prop. 2).

Prop. XXXIV. Theor.

If two planes which cut one another be each of them perpendicular to a third plane, their common section shall be perpendicular to the same plane.

Let two planes AB, BC (fig. 38) be each of them perpendicular to a third plane MN, and let BD be the common section of the first two: BD shall be perpendicular to the plane MN.

If it be not, from the point D draw, in the plane AB, the straight line DE at right angles to AD (I. 11) the common section of the planes AB and MN; and in the plane BC draw DF at right angles to CD the common section of the planes BC and MN.

And because the plane AB is perpendicular to the plane MN, and DE is drawn in the plane AB at right angles to AD, their com-

mon section, DE is perpendicular to the plane MN (Def. 6). In the same manner, it may be proved that DF is perpendicular to the plane MN. Wherefore, from the point D two straight lines stand at right angles to the third plane, upon the same side of it, which is impossible (Prop. 8): therefore, from the point D there cannot be any straight line at right angles to the plane MN, except BD the common section of the planes AB, BC: therefore BD is perpendicular to the plane MN: which was to be proved.

Cor. 1. When three planes are perpendicular to each other, the intersection of any two of these planes is perpendicular to the third plane; and the three intersections are perpendicular to each other.

Cor. 2. A plane perpendicular to two given planes which cut one another is perpendicular to their intersection. For each of the given planes is perpendicular to the third plane; consequently their intersection is perpendicular to that plane, and reciprocally, the third plane is perpendicular to their intersection.

Prop. XXXV. Theor.

Two planes which are perpendicular to a third plane, and which pass through two parallel straight lines not perpendicular to that plane, are parallel.

Let the two planes AQ, CS (fig. 39) be perpendicular to the plane MN, and pass through the parallel straight lines AB, CD which are not perpendicular to the planc MN: the planes AQ and CS are parallel.

Let PQ and RS be the sections of the planes AQ, CS with the plane MN. In the planes AQ, CS, draw AP, CR perpendicular to PQ, RS; then AP and CR are perpendicular to the plane MN (Def. 6), and are therefore parallel (Prop. 17). And because the straight lines BA, AP, meeting one another, are parallel to DC, CR which meet one another, the planes BAP and DCR are parallel (Prop. 26): which was to be proved.

Prop. XXXVI. Theor.

Two straight lines, not in the same plane, being given, a straight line which is perpendicular to them both may always be drawn; only one such line can be drawn; and this line is the shortest distance between the two straight lines.

Let AB, CD (fig. 40) be any two straight lines not in the same plane; a straight line which is perpendicular to both AB and CD may be drawn.

Through any point A, in the straight line AB, draw AE parallel to CD (I. 31), and let the plane MN pass through AB and AE.

From any point D, in CD, draw DF perpendicular to the plane MN (Prop. 5); and in this plane draw FG parallel to AE, meeting AB in G. Then, since GF and CD are each parallel to AE, they are parallel to one another (Prop. 19); and therefore GF is in the same plane as CD and DF. If, therefore, GH be drawn parallel to DF it will meet CD. GH is perpendicular to both AB and CD.

Because DF is perpendicular to the plane MN, GH parallel to DF is also perpendicular to that plane (Prop. 18), and therefore perpendicular to the straight lines AB and GF which meet it in that plane (Def. 3). And since GF is parallel to CD, and GH is perpendicular to GF, it is also perpendicular to CD (I. 28). GH, therefore, is perpendicular to both AB and CD.

Beside GH no other line can be drawn which is perpendicular to both AB and CD.

For, if possible, let KI be perpendicular to CD and also to AB. Draw IL parallel to AE, and KO parallel to DF. Then since IL and CD are each parallel to AE, IL is parallel to CD; and therefore IK which is perpendicular to CD, is also perpendicular to IL (I. 28). But by supposition IK is perpendicular to AB; it is therefore perpendicular to the plane MN passing through AB and IL. And because DF is perpendicular to the plane MN, KO which is parallel to DF is also perpendicular to the plane MN. Therefore from the same point K the two straight lines KO, KI are both perpendicular to the plane MN; which is impossible (Prop. 13).

Also HG is the shortest distance between CD and AB.

For take any other straight line as KI; and, as before, draw KO parallel to DF or HG. Then DF being perpendicular to the plane MN, KO, parallel to DF, is also perpendicular to MN (Prop. 18); and therefore KI is oblique to it. Consequently, KO is less than KI (Prop. 14). But HG is equal to KO (I. 34); therefore HG is less than KI. In the same manner HG may be shown to be less than any other line that can be drawn from CD to AB: consequently, HG is the least line that can be so drawn.

Therefore two straight lines not in the same plane being given, &c.: which was to be proved.

Prop. XXXVII. Theor.

If two planes intersect one another their inclination to each other is everywhere the same; that is from whatever point in the common intersection of two planes, straight lines be drawn perpendicular to it, one in each plane, the angle contained by these lines will be the same.

Let BA (fig. 41) be the common intersection of the two planes BAM, BAN; and let AM, AN, in the respective planes, be perpen-

dicular to BA at the point A, and BP, BQ perpendicular to BA at the point B: the angle MAN is equal to the angle PBQ.

Since MA and PB, in the plane BM, are perpendicular to BA, they are parallel (I. 28); and for the same reason NA and QB are parallel: consequently, the angle MAN is equal to the angle PBQ (Prop. 20): which was to be proved.

Prop. XXXVIII. Theor.

Dihedral angles formed by faces or planes whose angles of inclination to each other are the same, are equal to one another: and conversely, if two dihedral angles are equal, the angles of inclination of their faces are equal.

Let MABN, RCDT (fig. 42) be two dihedral angles formed by the planes AP, AQ and CS, CV, whose inclinations are the same: the angle MABN is equal to the angle RCDT.

Let DC be equal to AB; from A and B, let AM, BP be perpendicular to AB in the plane AP, and AN, BQ be perpendicular to it in the plane AQ; and from C and D, let CR, DS be perpendicular to DC in the plane CS, and CT, DV perpendicular to it in the plane CV.

The inclinations of the planes AP, AQ and CS, CV being the same, the angles MAN, RCT, PBQ, SDV are equal to one another. Now if the dihedral angle RCDT be applied to the dihedral angle MABN, so that DC coincides with BA, and DV, CT with BQ, AN; DS, CR will coincide with BP, AM, because the angle VDS is equal to the angle QBP, and the angle TCR to the angle NAM. DV and CT coinciding with BQ and AN, the plane DT, however extended, will everywhere coincide with the plane BN similarly extended; and DS and CR coinciding with BP and AM, the plane DR, however extended, will everywhere coincide with the plane BM similarly extended. Consequently the dihedral angles formed by the planes DT, DR, and BN, BM will coincide, and therefore be equal to one another.

Conversely, let the two dihedral angles MABN, RCDT be equal; the angle MAN is equal to the angle RCT.

For the dihedral angle RCDT being applied to MABN, as before, so that the plane DC coincides with the plane BN, and the straight line CD with AB, the plane DR must coincide with the plane BM, because the dihedral angles are equal; CR will coincide with AM; and the angle RCT will therefore coincide with the angle MAN, and be equal to it.

Therefore dihedral angles formed, &c.: which was to be proved.

Prop. XXXIX. Theor.

Dihedral angles are to one another as the angles of the inclinations of their faces.

Let MABN, PCDQ (fig. 43) be two dihedral angles; MA, NA perpendicular to BA, at the point A, in the planes BM, BN, which form the angle MABN; and PC, QC perpendicular to DC, at the point C, in the planes DP, CQ, which form the angle PCDQ. The angle MAN is therefore the inclination of the planes MB, NB; and the angle PCQ the inclination of the planes PD, QD. The angle MABN is to the angle PCDQ as the angle MAN to the angle PCQ.

At the point A, in the plane MAN, make any number of angles NAR, RAS each equal to the angle MAN; and at the point C, in the plane PCQ, make any number of angles QCT, TCU, UCV, each equal to the angle PCQ.

Then because BA is perpendicular to MA and NA, it is perpendicular to the plane MAN (Prop. 4); and therefore perpendicular to RA, SA which meet it in that plane (Def. 3): that is RA, SA are perpendicular to AB, in the planes RB, SB. Consequently NAR is the inclination of the planes NB, RB to each other, and RAS is the inclination of the planes RB, SB to each other (Def. 6). And because the angles MAN, NAR, RAS are all equal, the dihedral angles MABN, NABR, RABS are all equal (Prop. 38). Therefore whatever multiple the angle MAS is of the angle MAN, the same multiple is the dihedral angle MABS of the dihedral angle MABN.

In the same manner it may be shown that, whatever multiple the angle PCV is of the angle PCQ, the same multiple is the dihedral angle PCDV of the dihedral angle PCDQ.

If the angle MAS be equal to the angle PCV, the dihedral angle PCDV will be equal to the dihedral angle MABS (Prop. 38); if the angle MAS be greater than the angle PCV, the dihedral angle MABS will be greater than the dihedral angle PCDV; and if less, less. Therefore there are four magnitudes, the angles MAN, PCQ, and the dihedral angles MABN, PCDQ, and of the first and third the angle MAN and the dihedral angle MABN, any equimultiples whatever have been taken, namely, the angle MAS and the dihedral angle MABS; and of the second and fourth, the angle PCQ and the dihedral angle PCDQ, any equimultiples whatever have been taken, the angle PCV and the dihedral angle PCDV; and it has been shown that if the multiple of the first be equal to that of the second, the multiple of the third is equal to that of the fourth; if greater, greater; and if less, less: therefore (V. Def. 5) as the angle MAN is to the angle PCQ, so is the dihedral angle MABN to the dihedral angle PCDQ; which was to be proved.

Cor. 1. The angle of inclination of the faces of a dihedral angle is a measure of that angle.

Cor. 2. A dihedral angle is said to be *right, acute* or *obtuse*, according as the angle which measures it is right, acute or obtuse.

PROP. XL. THEOR.

When two parallel planes are cut by a third plane, the corresponding dihedral angles are equal.

Let the two parallel planes MN, PQ (fig. 44) be cut by the plane RS; the dihedral angles MABR, PCDR are equal.

Through A draw a plane perpendicular to AB (Prop. 10). Let the section of this plane with the plane RS be AR, and with the parallel planes NM, QP, the parallels AM, CP (Prop. 27). Since AM and CP are parallel, the angle MAR is equal to the angle PCR (I. 28). And because the plane RAM is perpendicular to AB, the angles MAB, RAB are right angles (Def. 3); and therefore the angle MAR is the inclination of the faces of the dihedral angle MABR (Def. 7). Again, because CD is parallel to AB (Prop. 27), CD is at right angles to the plane RAM or RCP (Prop. 18); the angles PCD, RCD are therefore right angles; and PCR is the inclination of the faces of the dihedral angle PCDR (Def. 7). And since the angles of inclinations MAR and PCR have been shown to be equal, the dihedral angles MABR and PCDR are also equal (Prop. 38): which was to be proved.

Scholium. The converse of the proposition is not true, that is, if the corresponding dihedral angles made by a plane cutting two planes are equal, these planes are not necessarily parallel. For if AB, CD (fig. 45) are two straight lines in the plane RS, which meet in the point O, and BM is a plane passing through AB, a plane DP may be drawn through CD which shall make an angle with the plane RS, equal to the angle which the plane BM makes with RS; and the planes DP and BM, if produced, will meet, for the point O is in each of these planes.

But if besides the equality of the dihedral angles the straight lines AB and CD are parallel, then the planes BM and DP will be parallel.

PROP. XLI. THEOR.

To bisect a dihedral angle.

Let the dihedral angle which it is required to bisect be MABN (fig. 46), whose faces are BM and AN.

From any point C in the intersection AB of the faces draw CQ in the plane MB, and CR in the plane NB, at right angles to AB (I. 11). In the plane QCR draw CS bisecting the angle QCR (I. 9);

and through BA and CS let the plane BP pass: the plane BP will bisect the dihedral angle MABN.

Because BA is at right angles to QC and RC, it is at right angles to the plane QCR (Prop. 4); and therefore to the straight line SC in that plane. QC, SC, RC being, therefore, at right angles to AB, the angles QCS, RCS are the inclinations of the plane BP to the planes BM, BN (Def. 7); and since these angles are equal, the dihedral angle MABP is equal to the dihedral angle NABP (Prop. 38), and the dihedral angle MABN is bisected by the plane BP: which was required to be done.

Prop. XLII. Theor.

Every point in a plane which bisects a dihedral angle is equally distant from the faces of that angle; and every point which is without the bisecting plane is unequally distant from those faces.

Let the plane BP bisect the dihedral angle MABN (fig. 47): any point C in the plane BP is equally distant from the faces BM, BN.

From C draw CD, CE perpendiculars to the planes BM, BN (Prop. 5); and through C, D, E let the plane CDE pass, cutting the planes BM, BN, BP in DF, EF, CF. The plane CDE, being perpendicular to the planes BM, BN (Prop. 28), is perpendicular to AB (Prop. 34); therefore FD, FC, FE are at right angles to BA (Def. 3). The angles DFC, EFC are therefore the inclinations of the plane BP to the planes BM, BN (Def. 7). And since the dihedral angles MABP, NABP are equal, the angles of inclination DFC, EFC are also equal (Prop. 38); and the right angle CDF is equal to the right angle CEF; and the side FC opposite to equal angles in the two triangles CDF, CEF is common to both; therefore CD is equal to CE (I. 26), that is (Prop. 9. Schol.), the point C in the plane BP is equally distant from the faces BM, BN of the dihedral angle MABN.

Next let G be a point without the plane BP; G is unequally distant from the planes BM and BN.

From G let fall GH, GE perpendiculars to the planes BM, BN (Prop. 5); and through H, G, E let the plane HGE pass, cutting the planes BM, BP, BN in HF, CF, EF. From C, the intersection of GE with the plane BP, draw CD parallel to GH, and therefore perpendicular to the plane BM (Prop. 18).

Because GH is perpendicular to the plane BM, GH is less than GD (Prop. 9). And because GC and CD are greater than GD (I. 20), and that CD is equal to CE; GE is greater than GD. But GH is less than GD; much more then is GE greater than GH.

Therefore every point in a plane, &c.: which was to be proved.

Def. (a). *The complement of an angle* is what it wants of a right angle.

Def. (b). *The supplement of an angle* is what it wants of two right angles.

Prop. XLIII. Theor.

The angle formed by perpendiculars let fall from a point within a dihedral angle, upon its faces, is the supplement of the measure of the dihedral angle.

Let OC, OD (fig. 48) be perpendiculars from the point O within the dihedral angle MABN, on its faces MB, NB; the angle COD is the supplement of the inclination of the faces MB, NB.

Through C, O, D let a plane pass, cutting the planes MB, NB in CE, DE; then, as in the last proposition, the angle CED is the measure of the dihedral angle MABN. And because the four angles of the quadrilateral CEDO are together equal to four right angles (I. 32); and that the angles OCE, ODE are right angles; the angles COD and CED are equal to two right angles: that is, the COD is the supplement of the angle CED which measures the dihedral angle MABN: which was to be proved.

Prop. XLIV. Theor.

If a solid angle be contained by three plane angles, any two of them are together greater than the third.

Let the solid angle at A (fig. 49) be contained by the three plane angles BAC, CAD, DAB: any two of them together shall be greater than the third.

If the angles BAC, CAD, DAB be all equal, it is evident that any two of them are together greater than the third. But if they are not, let BAC be that angle which is not less than either of the other two, and is greater than one of them DAB; and at the point A in the straight line AB, make, in the plane which passes through BA, AC, the angle BAE equal to the angle DAB (I. 23); and make AE equal to AD, and through E draw BEC cutting AB, AC, in the points B, C; and join DB, DC.

And because DA is equal to AE, and AB is common, the two DA, AB are equal to the two EA, AB, each to each; and the angle DAB is equal to the angle EAB: therefore the base DB is equal to the base BE (I. 4). And because DB, DC are greater than CB (I. 20), and one of them BD has been proved equal to BE a part of CB, therefore the other DC is greater than the remaining part EC (I. Ax. 3). Again, because DA is equal to AE, and AC common, but the base DC greater than the base EC; therefore the angle DAC is greater than the angle EAC (I. 25); and, by the construction, the angle DAB is equal to the angle BAE; wherefore the angles DAB, DAC are together greater (I. Ax. 4) than BAE, EAC, that is, than the angle BAC: but BAC is not less than either of the angles DAB, DAC: therefore BAC with either of them is greater than the other. Wherefore, if a solid angle, &c.: which was to be proved.

Prop. XLV. Theor.

Every solid angle is contained by plane angles, which together are less than four right angles.

First, let the solid angle at A (fig. 50) be contained by three plane angles BAC, CAD, DAB: these three together shall be less than four right angles.

Take in each of the straight lines AB, AC, AD, any points B, C, D, and join BC, CD, DB. Then because the solid angle at B is contained by the three plane angles CBA, ABD, DBC, any two of them are greater than the third (Prop. 44); therefore the angles CBA, ABD are greater than the angle DBC: for the same reason, the angles BCA, ACD are greater than the angle DCB; and the angles CDA, ADB greater than BDC: wherefore the six angles CBA, ABD, BCA, ACD, CDA, ADB are greater than the three angles DBC, BCD, CDB: but the three angles DBC, BCD, CDB are equal to two right angles (I. 32); therefore the six angles CBA, ABD, BCA, ACD, CDA, ADB are greater than two right angles: and because the three angles of each of the triangles ABC, ACD, ADB are equal to two right angles, therefore the nine angles of these three triangles, viz. the angles CBA, BAC, ACB, ACD, CDA, DAC, ADB, DBA, BAD, are equal to six right angles: of these the six angles CBA, ACB, ACD, CDA, ADB, DBA are greater than two right angles; therefore the remaining three angles BAC, CAD, DAB, which contain the solid angle at A, are less than four right angles.

Next, let the solid angle at A (fig. 51) be contained by any number of plane angles BAC, CAD, DAE, EAF, FAB: these shall together be less than four right angles.

Let the planes in which the angles are be cut by a plane, and let the common sections of it with those planes be BC, CD, DE, EF, FB. And because the solid angle at B is contained by three plane angles CBA, ABF, FBC, of which any two are greater than the third (Prop. 44), the angles CBA, ABF are greater than the angle FBC: for the same reason, the two plane angles at each of the points C, D, E, F, viz. those angles which are at the bases of the triangles having the common vertex A, are greater than the third angle at the same point, which is one of the angles of the polygon BCDEF: therefore all the angles at the bases of the triangles are together greater than all the angles of the polygon: and because all the angles of the triangles are together equal to twice as many right angles as there are triangles (I. 32); that is, as there are sides in the polygon BCDEF; and that all the angles of the polygon, together with four right angles, are likewise equal to twice as many right angles as there are sides in the polygon (I. 32. Cor. 1); therefore all the angles of the triangles are equal to all the angles of the

polygon together with four right angles (I. Ax. 1): but all the angles at the bases of the triangles are greater than all the angles of the polygon, as has been proved; wherefore the remaining angles of the triangles, viz. those at the vertex, which contain the solid angle at A, are less than four right angles. Therefore, every solid angle, &c.: which was to be proved.

Prop. XLVI. Theor.

If from a point taken within a trihedral angle perpendiculars are let fall upon the faces of the angle, that is upon the planes which form it, these perpendiculars will be the edges or intersections of the planes of a new trihedral angle, the plane angles forming which are the supplements of the measures of the dihedral angles at the edges about the original solid angle; and the measures of the dihedral angles about which are the supplements of the plane angles forming that solid angle.

From the point O (fig. 52) within the trihedral angle formed by the three plane angles ASB, ASC, BSC, let the perpendiculars OP, OQ, OR upon the faces ASB, ASC, BSC be drawn; these perpendiculars are the edges of a trihedral angle, the plane angles of which, POQ, POR, QOR, are respectively the supplements of the measures of the dihedral angles at the edges SA, SB, SC, about the solid angle at S.

Let PT, QT be the intersections of the plane POQ with the planes ASB, ASC; PV, RV the intersections of the plane POR with the planes ASB, BSC; and RU, QU the intersections of the plane QOR with the planes BSC, ASC.

Then since OP and OQ are perpendiculars let fall from the point O upon the planes ASB, ASC which are the faces of the dihedral angle whose edge is SA, the angle POQ, one of the plane angles forming the solid angle at O, is the supplement of the angle PTQ which measures the corresponding dihedral angle about the solid angle at S (Prop. 43).

In the same manner it may be shown that the other plane angles POR, QOR, forming with POQ the solid angle at O, are the supplements of PVR, QUR which measure the other dihedral angles about the solid angle at S.

Also, the measures of the dihedral angles at the edges OP, OQ, OR, about the solid angle at O, are respectively the supplements of the plane angles ASB, ASC, BSC forming the solid angle at S.

Since OP is at right angles to the plane ASP, it is at right angles to TP and PV (Def. 3), and therefore the angle TPV measures the dihedral angle whose faces are the planes OPTQ, OPVR, and whose edge is PO. And because ST and SV are perpendiculars from the point S upon the planes POQ, POR (Prop. 34), which are the faces

of the dihedral angle whose edge is OP, the angle TSV is the supplement of the angle TPV, which measures the dihedral angle TPOV whose edge is OP (Prop. 43); and consequently the measure of the dihedral angle TPOV is the supplement of the angle ASB.

In the same manner it may be shown, that the angle TQU, which measures the dihedral angle TQOU whose edge is OQ, is the supplement of the angle ASC; and that the angle URV, which measures the dihedral angle VROU whose edge is OR, is the supplement of the angle BSC.

Therefore, if from a point taken within a trihedral angle, &c.: which was to be proved.

Scholium. On account of these properties the trihedral angles whose vertices are S and O are called supplemental to each other.

Prop. XLVII. Theor.

If two trihedral angles have the three faces or plane angles of the one equal to the three faces of the other, each to each; the dihedral angles contained by the faces of the one shall be equal to the dihedral angles contained by the faces equal to them of the other.

Let the trihedral angles at S and V (fig. 53) have the three faces ASB, ASC, BSC, at S, equal to the three faces DVE, DVF, EVF, at V, each to each; the dihedral angle contained by the faces ASB and ASC shall be equal to the dihedral angle contained by the faces DVE and DVF; the angle contained by ASB and BSC equal to the angle contained by DVE and EVF; and the angle contained by ASC and BSC equal to the angle contained by DVF and EVF.

In SA, SB, SC and VD, VE, VF take SG, SH, SI and VK, VL, VM all equal to each other; join GH, HI, IG and KL, LM, MK. And because SG is equal to SH, the angle SHG is equal to the angle SGH (I. 5), and therefore each of these angles is less than a right angle (I. 17). In the same manner it may be shown, that each of the angles SGI, SIG is less than a right angle; and that each of the angles VKL, VKM is less than a right angle. In GS take any point N, and from it, in the plane ASB, draw NP at right angles to SA, meeting GH, or GH produced (because the angles NGH, GNP are less than two right angles) in P; and in the plane ASC draw NO at right angles to SA, meeting GI, or GI produced (I. Ax. 12) in O: join OP. In KV take KQ equal to GN, and from Q draw, in the planes DVE, DVF, QT, QR at right angles to VD, meeting KL, KM, or these produced (I. Ax. 12) in T, R: and join RT.

Because in the two triangles GSH, KVL, the two sides GS, SH are equal to the two KV, VL, each to each, and their contained angles GSH, KVL are equal (by hypothesis), the base GH is equal to the base KL, and the angles SGH, SHG equal to the angles

VKL, VLK, each to each (I. 4). In like manner it may be shown that GI is equal to KM, the angles SGI, SIG equal to VKM, VMK, each to each; that IH is equal to ML, and the angles SIH, SHI equal to VML, VLM, each to each. And because, in the two triangles PNG, TQK, the angles PNG, PGN are equal to TQK, TKQ, each to each, and GN is equal to KQ, GP is equal to KT and NP to QT (I. 26). In the same manner it may be shown, in the two triangles NGO, QKR, that GO is equal to KR, and NO to QR. Again, in the two triangles HGI, LKM, the sides HG, GI are equal LK, KM, each to each, and IH is equal to ML, therefore the angle HGI is equal to the angle LKM (I. 8). And because, in the two triangles PGO, TKR, the sides PG, GO are equal TK, KR, each to each, and the angle PGO has been shown to be equal to the angle TKR, the base PO is equal to the base TR (I. 4); and it has been shown that the sides PN, NO in the triangle PNO are equal to the sides TQ, QR in the triangle TQR, therefore the angle PNO is equal to the angle TQR (I. 8). But the angle PNO is the inclination of the planes ASB and ASC, because NP and NO are drawn, in these planes, perpendicular to SA (Def. 7); and for the same reason the angle TQR is the inclination of the planes DVE, DVF; therefore the inclination of the planes ASB, ASC is equal to the inclination of the planes DVE, DVF, that is, the dihedral angle contained by the faces ASB, ASC is equal to the dihedral angle contained by the faces DVE, DVF (Prop. 38).

In like manner it may be shown, that the dihedral angle contained by the faces ASB, BSC is equal to that contained by the faces DVE, EVF, and the angle contained by the faces ASC, BSC equal to that contained by the faces DVF, EVF.

Wherefore if two trihedral angles, &c.: which was to be proved.

Scholium. The demonstration is precisely the same whether the faces which are equal follow the same order, in the same direction, about the two solid angles, or not; that is, supposing the intersection SC of the two faces ASC, BSC to be in front of the face ASB, whether the intersection VF of the two faces DVF, EVF be in front of the face DVE, or behind it. In the former case, the solid angles at S and V are *equal,* because if the solid angle at S be applied to the solid angle at V so that the face ASB coincide with the face DVE, the faces ASC and BSC will coincide with the faces DVF and EVF, the dihedral angles contained by the former faces with ASB being equal to those contained by the latter faces with DVF, and also in the same direction; in the latter case the solid angles at S and V are only *symmetrical,* because their faces will not coincide, the dihedral angles though equal not being in the same direction.

Cor. It follows from this and Prop. 46, that if the three dihedral angles about two trihedral angles be equal, each to each, the faces or plane angles which form the trihedral angles will be equal, each

to each. For the faces in the two supplemental trihedral angles being the supplements of the measures of the dihedral angles about the original trihedral angles (Prop. 46) will be equal, each to each; and therefore by this proposition the dihedral angles contained by these faces will be equal, each to each; consequently the faces of the original trihedral angles, which are the supplements of the measures of these dihedral angles, will be equal, each to each.

Prop. XLVIII. Theor.

If two trihedral angles have two faces or plane angles of the one, equal to two faces of the other, each to each, and the dihedral angle contained by the two faces in the one equal to the dihedral angle contained by the two faces equal to them of the other; they shall have their third faces equal, and their remaining dihedral angles shall be equal, each to each, namely those to which the equal faces are opposite.

Let the trihedral angles at S and V (fig. 53) have the two faces ASB, ASC equal to the faces DVE, DVF, each to each, and the dihedral angle contained by the faces ASB, ASC equal to the dihedral angle contained by the faces DVE, DVF; they shall have their third faces BSC, EVF equal; the dihedral angle contained by the faces ASB, BSC shall be equal to that contained by the faces DVE, EVF; and the dihedral angle contained by ASC, BSC equal to that contained by DVF, EVF.

The same construction being made as in the last proposition, it may be shown as before, that GH is equal to KL, and the angle SGH equal to the angle VKL; that GI is equal to KM, and the angle SGI equal to the angle VKM; that GP and NP are equal to KT and QT, each to each; and that GO and NO are equal to KR and QR, each to each. And because the dihedral angle contained by the faces ASB, ASC is equal to that contained by the faces DVE, DVF, the inclination of the former planes, the angle PNO, is equal to the inclination of the latter planes, the angle TQR (Prop. 38); therefore in the two triangles PNO, TQR, the sides PN, NO are equal to TQ, QR, each to each, and the angle PNO is equal to the angle TQR; therefore the base PO is equal to the base TR (I. 4). Hence in the two triangles PGO, TKR the three sides in the one are equal to the three sides in the other, each to each, and therefore the angle PGO is equal to the angle TKR (I. 8); consequently, in the two triangles HGI, LKM, the two sides HG, GI are equal to the two LK, KM, each to each, and the angle HGI is equal to the angle LKM, and therefore the base HI is equal to the base LM (I. 4). Since then, in the two triangles HSI, LVM, the two sides HS, SI are equal to the two LV, VM, each to each, and the base HI has been shown equal to the base LM, the angle HSI is

equal to the angle LVM; that is, the third face of the trihedral angle at S is equal to the third face of the trihedral angle at V.

Since the three faces of the trihedral angle at S are equal to the three faces of that at V, their remaining dihedral angles are equal (Prop. 48).

Wherefore, if two trihedral angles, &c.: which was to be proved.

Cor. From this and Prop. 46, it follows that, if two trihedral angles have a face, or plane angle, in the one equal to a face in the other, and the dihedral angles adjacent to these faces equal, each to each, their other faces will be equal, each to each, and their remaining dihedral angles also equal. For in this case, the two supplemental trihedral angles will have two faces in the one equal to two faces in the other, each to each, and the dihedral angles contained by these faces equal (Prop. 46); therefore, by this proposition, they will have their third faces equal, and their remaining dihedral angles equal, each to each; and consequently the faces in the original trihedral angles being the supplements of these dihedral angles (Prop. 46), will be equal, each to each, and the third dihedral angles in the original trihedral angles, which are the supplements of the third faces in the supplemental angles, are also equal.

THEOREMS TO BE DEMONSTRATED.

1. If through a point in a given straight line several equal straight lines be drawn, making equal angles with the given line, the extremities of these straight lines will be in the circumference of a circle of which the plane is perpendicular to the given straight line, and of which the centre is in this line.

2. When a straight line meeting three straight lines in a plane makes equal angles with them, these angles are right angles, and the straight line is perpendicular to the plane.

3. All the parallels to a given straight line, drawn through different points of any straight line, are in the same plane.

4. If, on the same side of a plane, equal and parallel straight lines be drawn from different points of a straight line in the plane, the extremities of these parallels will be in a straight line parallel to the plane.

5. If a straight line be perpendicular to a plane, every plane parallel to the line will be perpendicular to that plane.

6. If two planes be perpendicular to each other, every straight line perpendicular to one of them will be either parallel to the other, or be wholly contained in it.

And, conversely, if a straight line be parallel to a plane, every plane perpendicular to this line will also be perpendicular to the former plane.

7. If a plane be parallel to two straight lines which cut one another, it will be parallel to the plane of these lines.

8. The triangles formed by joining the corresponding extremities of three equal and parallel straight lines, not all in the same plane, are equal, and their planes are parallel.

9. If, from different points of a plane, equal and parallel straight lines be drawn on the same side of it, their extremities will be in a plane parallel to the former.

10. If a straight line be parallel to a plane, every straight line parallel to that line will be parallel to the plane, or be wholly contained in it.

11. Two straight lines respectively parallel to two parallel straight lines are parallel to each other.

12. Two planes respectively parallel to two parallel planes are parallel to each other.

13. If two planes be respectively parallel to two planes which cut one another, the intersection of the former planes will be parallel to that of the latter.

14. A straight line which is parallel to one of two parallel planes is parallel to the other, or is wholly contained in it.

15. Planes respectively perpendicular to two straight lines which intersect, likewise intersect.

16. When a straight line meets a plane, any straight line perpendicular to the plane will meet any plane perpendicular to the former straight line.

17. Planes perpendicular to the sides of a triangle at their middle points intersect each other, in the same straight line.

18. Parallel straight lines which meet a plane make equal angles with it.

19. Parallel planes which meet a straight line make equal angles with it.

20. If the faces of two dihedral angles be parallel, they will be equal when their faces are both towards the same or towards contrary parts; and one will be the supplement of the other when two of their faces are towards the same parts, and the other two towards contrary parts.

21. The planes which bisect the three dihedral angles about a trihedral angle intersect in the same straight line.

22. If two straight lines not in the same plane be divided in the same ratio, three parallel planes may be drawn, whereof two pass through the corresponding extremities of the straight lines, and the third through the points of section.

THE

ELEMENTARY PRINCIPLES

OF

DESCRIPTIVE GEOMETRY.

1. Descriptive Geometry has been defined, "The science which teaches methods of representing accurately geometrical magnitudes, and how to perform graphically all possible operations upon these magnitudes*." By means of it questions which embrace the three dimensions of space are reduced to constructions which may be effected on a plane. Although solid geometry and descriptive geometry treat of the same magnitudes, they differ essentially from each other. In the former, magnitudes in space are in general represented rather arbitrarily, according to the appearance they may present as viewed on a plane, and although the conditions arrived at are not affected by want of accuracy in the representations, we have here no means of construction which will give the real dimensions of the magnitudes represented; the latter affords these means.

To describe a sphere that shall circumscribe a triangular pyramid, is the same problem with regard to space, which the description of a circle about a triangle is with regard to a plane. In the latter problem, the perpendiculars to two of the sides of the triangle at their middle points, will intersect in a point, and the perpendicular to the third side, at its middle point, must meet the two other perpendiculars in their intersection: in the former problem, the planes perpendicular to each pair of the edges of one of the solid angles, at their middle points, will meet in a line; the three lines of intersection will meet in a point; and in the same point must meet, the intersections of the planes drawn perpendicular to the other edges of the pyramid, at their middle points. In the case of the circle, all the operations can be effected, and the lines, both those given and those required in the construction, can be correctly represented in their true dimensions, because they are all in the same plane. In the sphere the case is very different: the lines given, not being

* Vallée, Traité de la Géométrie Déscriptive, Introduction, p. xvi.

in the same plane, are not represented in a figure on a plane either in their true positions or their true dimensions; and the same is the case with respect to the planes and lines required in the construction: the requisite operations cannot be performed on paper, that is, on a plane. The one is a real construction which can be effected by means of a ruler and a pair of compasses; the other is an ideal construction which can only thus be really effected in space, where the solid, the several planes and their intersections exist in their true positions and dimensions. Descriptive geometry furnishes the means of actually effecting such constructions by referring points, lines, and planes as they exist in space to two fixed planes, by means of perpendiculars to these planes.

2. The most simple manner of determining the position of a point in a plane is by referring it to two fixed straight lines in the plane, at right angles to each other, by means of perpendiculars let fall from it upon these lines: the position of the point will be determined when the lengths of these perpendiculars are known.

3. In the same manner, if a point above the plane in which are the two fixed straight lines, be referred to that plane by a perpendicular let fall from it on the plane, and the foot of the perpendicular be referred to the fixed lines by perpendiculars, the position of the point in space will be determined when the lengths of the perpendiculars on the fixed lines, and that of the perpendicular on the plane are known. If every point whose position in space is required, is thus referred to a plane and to two fixed lines in it, the positions of these points would be determined.

If, for example, all the points to be determined in a geometrical magnitude which is above a horizontal plane, be referred to the plane by perpendiculars to it, that is, by vertical lines, and the intersections of these vertical lines with the horizontal plane be referred to two fixed lines in it, by perpendiculars on them, the positions of the points in space will be determined by the lengths of the perpendiculars on the fixed lines, and of the perpendiculars on the plane, drawn from the several points; the positions of the points in the horizontal plane giving what is termed the *Plan* of the points in the body, and the perpendiculars giving the heights of these points above the plane. In this mode of representation, which is called *Horizontal Projection,* in order to give a clear idea of the positions in space of the original points, it is necessary that the height of each point above the horizontal plane should be indicated in connexion with its projection or representation on that plane. When the number of points to be thus represented is considerable, the plan maybecome complicated by the figures or lines indicating the heights; but still this method, to which we shall hereafter recur, will be advantageously employed when most of the data and the results to be obtained from them are numerical. An equivalent method is, how-

ever, better adapted for representing magnitudes in their true dimensions, and also in their geometrical relations.

4. A point in space being, as before, referred, by a perpendicular, to the plane in which the two fixed straight lines are, which we will suppose to be the plane of the paper, and, for the sake of illustration, we will in the first instance consider this to be vertical—so that the figure is viewed as drawings usually are—one of the fixed lines being horizontal, and the other vertical; and from the foot of this perpendicular there being drawn a perpendicular to the horizontal fixed line; if, in the same manner, the point be referred, by a perpendicular, to another plane passing through the horizontal fixed line, at right angles to the paper, which plane will therefore be horizontal, and from the foot of this perpendicular in the horizontal plane a perpendicular be let fall on the intersection of the two planes; the position of the point will be determined by three lines, the lengths of the perpendiculars let fall in the two planes on their intersection, and the distance of these perpendiculars from the intersection of the two fixed lines. We have supposed the vertical plane to be represented on the paper, and therefore the lines in the vertical plane will be represented on the paper in their true dimensions and positions: if now we conceive the plane of the paper to revolve about the intersection of the vertical and horizontal planes, until it coincides with the latter, the lines in the horizontal plane will likewise be represented on the paper in their true dimensions and positions. This is the fundamental principle of Descriptive Geometry, and if well understood little difficulty will be found in its practical application. We will therefore endeavour to render what has been here stated more clear, by means of figures.

5. The point A (Plate II. fig. 1), in the plane xu, being referred to two fixed lines xy, xv, in the plane, by perpendiculars Am, An, let fall from A upon these lines, the position of the point A will be determined when the lengths of the perpendiculars Am, An are known.

6. If the point A (fig. 2), supposed to be in front of the plane xu, be referred to that plane, by a perpendicular Aa', let fall from A upon xu, and the point a' be referred to xy, xv, by the perpendiculars $a'm$, $a'n$, the position of the point A in space is determined when the lengths of the perpendiculars Aa', $a'm$, $a'n$ are known. The length of the line Aa' is not however correctly represented in the figure, nor is its position with reference to $a'm$ and $a'n$ so represented.

7. If the point A, besides being referred, by the perpendicular Aa', to the plane xu, which we will suppose vertical, be referred, by a perpendicular Aa, to the plane xz at right angles to the plane xu, and therefore horizontal; and if from a, am be drawn perpendicular to xy, the position of the point A will be determined by the two perpendiculars on xy, ma' and ma, and mx, or $a'n$ perpendicular to xv. The line Aa, or its equal ma', will represent correctly the distance of the

point A from the plane xz, but then according to the ideal view o the lines in the figures, Aa' or ma will not so represent the distance of A from the plane. If, however, we conceive the plane of the paper to revolve about the line xy, the intersection of the vertical and horizontal planes, until it coincides with the latter, and the foot of the perpendicular Aa on the horizontal plane to be represented on the paper in its true position, and the line ma in its true direction, that is, at right angles to the line xy, as in fig. 3, without any regard to the ideal representation in fig. 2, the lines xm, ma', ma will be represented in their true dimensions.

We have here supposed that the plane of the paper revolves about the horizontal line xy until it become horizontal, and that in this position of the paper the point a, where the perpendicular on the horizontal plane, from the point A, falls on it, is marked; but it is evident that the result will be the same, if we suppose that the paper is horizontal, that the horizontal plane is represented on it by xz, that the point A above it is referred to this plane by a perpendicular to it from A, meeting the plane in a; and the point A being referred, by a perpendicular, to the vertical plane, this plane be conceived to revolve about the horizontal line xy until it coincides with the plane of the paper. This last is the usual way of presenting the principle on which are founded the methods of Descriptive Geometry. We now proceed to the details of these methods.

Definitions.

8. When a point without a plane is referred to the plane by a perpendicular to it, the point where the perpendicular from the original point meets the plane is called the *Orthographic Projection* of the original point.

Thus the point a (fig. 4), where the perpendicular Aa, from the point A, to the plane xz meets the plane, is called the orthographic projection of the point A on the plane xz.

9. The line Aa which determines the projection a of the point A is called *the Projecting straight line*, or simply, the *Projecting Line* of the point A.

10. If from the several points A, M, N, O . . . B of a straight line AB (fig. 5), perpendiculars Aa, Mm, Nn, Oa . . . Bb be let fall upon the plane xz, the line $amnob$ will be the *orthographic projection* of AB on the plane xz.

Since the plane passing through Aa and AB will contain all the perpendiculars Mm, Nn, Oo . . . Bb (Geometry of Planes, Prop. 33), the orthographic projection of a straight line AB is a straight line ab determined by the projections of its extremities.

11. The plane $ABba$ passing through the straight line AB, perpendicular to the plane xz, is called the *Projecting Plane* of that line.

12. If from the several points A, M, N, O . . . B (fig. 6) of *any* line AMNOB perpendiculars Aa, Mm, Nn, Oo . . . Bb be let fall upon the plane xz, the line *amnob* passing through all the points a, m, n, o, . . . b will be the orthographic projection of the line AMNOB on the plane xy.

13. The surface which contains all the lines Aa, Mm, Nn, &c., and which coincides with the curve AMNOB, is the projecting surface of that curve upon the plane xz.

14. The plane upon which points and lines are projected is called the *Plane of Projection*.

Thus, xz in the foregoing cases is the plane of projection.

15. The point in which a straight line, produced if necessary, meets a plane is called its *Trace* upon that plane.

Thus, let the straight line AB (fig. 7), when produced, meet the plane xz in the point α; the point α is the trace of AB on the plane xz.

16. The straight line in which a plane, or the line in which any surface, produced if necessary, meets a plane is called its *Trace* upon that plane.

Thus, αp (fig. 8) being the line in which the plane AB$p\alpha$ meets the plane xz, αp is the trace of AB$p\alpha$ on the plane xz.

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17. It is evident that points and lines may, in the foregoing manner, be projected on two or more planes by perpendiculars let fall upon these planes. As for our present purpose more than two planes are not necessary, we shall restrict ourselves to these.

Aa (fig. 9) being perpendicular to the plane xz, a is the projection of the point A upon xz; and if Aa' be perpendicular to the plane xu which cuts the plane xz in xy, a' will be the projection of A upon xu.

Aa and Bb (fig. 10) being perpendicular to the plane xz, the straight line ab is the projection of the straight line AB upon the plane xz; and if Aa' and Bb' be perpendicular to the plane xu, the straight line $a'b'$ will be the projection of the straight line AB upon the plane xu.

When two planes xz, xu are thus employed, they are called *Coordinate Planes*.

18. A point is completely determined when its projections on two planes which cut one another are given.

a and a' (fig. 9) being the projections of a point upon the planes xz and xu, the point of which they are the projections must be A, the intersection of the perpendiculars aA, a'A to the plane xz, xu drawn from the points a, a'.

It is to be remarked here, that two points, taken at will on two planes which cut one another, may not be the projections upon those planes of any single point in space.

19. In order that two points, which are respectively on two planes that cut one another, may be the projections of the same point in space, it is necessary that the perpendiculars drawn from these points upon the intersection of the two planes, should fall upon the same point of this intersection, and this condition is sufficient.

Let a and a' (fig. 9) be two points on the planes xz and xu respectively; in order that they may be the projections of the same point in space, the perpendiculars am, $a'm$ from them, on the intersection xy of the planes, must fall at the same point m.

For if A be any point in space, the plane Aama' passing through its orthographic projections on the planes xz, xu is perpendicular to each of these planes (Prop. 32); consequently, each of the planes xz, xu being perpendicular to the plane Aama', their intersection xy is perpendicular to the plane Aama' (Prop. 34); and therefore each of the angles xma, xma' is a right angle: that is, the perpendiculars to the intersection xy drawn from the projections a, a' of the point A fall on the same point m of the intersection.

And this condition is sufficient.

Let am and $a'm$, drawn from the points a and a', in the planes xz and xu, perpendicular to xy, fall on the same point, m, of xy; a and a' are the orthographic projections of the same point in space.

For xy being perpendicular to ma and ma', it is perpendicular to the plane ama' (Prop. 4); therefore each of the planes xz, xu is perpendicular to the plane ama' (Prop. 32); and consequently the plane ama' is perpendicular to each of the planes xz, xu. If, therefore, straight lines be drawn from the points a, a' perpendicular to the planes xz, xu respectively, they will be in the plane ama' (Prop. 33), and will meet in a point A in that plane.

20. A straight line is determined in position and magnitude when its projections on two planes which cut each other are given.

Let ab, $a'b'$ (fig. 10) be the projections of a straight line on the planes xz, xu; the position and magnitude of the line of which ab, $a'b'$ are the projections, are determined.

Let a plane perpendicular to xz pass through ab, and be terminated by perpendiculars to xz, from a and b: and another plane perpendicular to xu pass through $a'b'$, and be terminated by perpendiculars to xu from a' and b': the intersection AB of the planes thus determined is the straight line of which ab, $a'b'$ are the projections.

21. In order that ab, $a'b'$ may be the projections of the same determinate straight line in space, it is necessary that a and a' should be the projections of one point in space, and also that b and b' should be the projections of another point in space; and, therefore, that the perpendiculars drawn on xy from a and a' should fall on the same point in it, and likewise that the perpendiculars on xy from b and b' should fall on the same point.

22. A straight line in space is, however, given in position, when

the positions of the projections of any two points of that line are given on one plane of projection, and the positions of the projections of any two points of the same line, whether the same points as the former two or not, are given on the other plane of projection.

For the straight line will be in the plane passing through the first two points, perpendicular to the first plane of projection, and also in the plane passing through the second two points, perpendicular to the second plane of projection: its position will therefore be determined by the intersection of these two planes.

23. When the two straight lines ab, $a'b'$ (fig. 11) are perpendicular to xy, and cut this line in different points m, n, they cannot be the projections of the same straight line. For through am let the plane amp pass, perpendicular to the plane xz, and through $a'n$ the plane $a'nq$, perpendicular to the plane xu. Then since the plane amp is perpendicular to the plane xz, and that xy is perpendicular to am, the common intersection of the two planes, xy is perpendicular to the plane amp (Def. 6), and the plane amp is perpendicular to xy. In the same manner it appears that the plane $a\,nq$ is perpendicular to xy. The two planes amp, $a'nq$ being perpendicular to the same straight line xy are parallel to each other (Prop. 25); and consequently cannot contain any straight line common to both.

24. When the two straight lines ab, $a'b'$ (fig. 12) perpendicular to xy meet this line in the same point m, then xy is perpendicular to the plane ama' (Prop. 4), and therefore the plane ama' is perpendicular to each of the planes xz, xu (Prop. 32). Hence the directions of the projections of every line in the plane ama' will be in the lines ma, ma'; and if only these lines be given, the direction in the plane ama' of the line of which they may be the projections will be undetermined: in order that it may in this case be determined, it is necessary that its projection $a''n$ on some other plane txs, not passing through xy, should be given. But if a, a' the projections of one extremity of the line in the plane ama' be given, and likewise b, b' the projections of the other extremity, the line AB of which ab, $a'b'$ are the projections will be given in magnitude and direction.

25. In general, the projections of any curve upon two planes which cut one another determine this curve.

Thus the curves abc and $a'b'c'$ (fig. 13) being the projections of the curve ABC upon the planes xz, xu, conceive perpendiculars to the respective planes through the several points of the curves abc, $a'b'c'$: they will form cylindric surfaces abcCAB, $a'b'c'$CAB, which must contain the curve ABC, and which consequently will determine this curve by their intersection.

If the cylindric surfaces do not cut each other, the given projections do not belong to the same curve in space.

26. A plane is determined when its traces upon two planes are given.

For two straight lines which cut one another determine the position of the plane passing through them (Prop. 2).

In general, the plane to be determined will cut the intersection xy of the two co-ordinate planes xz, xu; and it is evident that the point of meeting α (fig. 14) will be common to its two traces $a\alpha$, $a'\alpha$.

If the plane be parallel to xy, its traces $a\alpha$, $a'\alpha'$ (fig. 15) will also be parallel to xy (Prop. 23).

If the plane be perpendicular to xy, its traces will be perpendicular to xy (Def. 3).

If the plane be parallel to one of the co-ordinate planes, its trace upon the other plane will be parallel to xy (Prop. 27), and then this trace determines the plane.

If the plane pass through xy, its traces upon the two co-ordinate planes will be confounded in this line, which is not sufficient to determine the plane. In this case it is necessary to have its trace bx upon another co-ordinate plane sxt (fig. 16).

27. The preceding principles show how the data in problems in which the three dimensions of space are considered, may be represented accurately by points and lines situate in two fixed planes which cut each other, and which are inclined to each other at a convenient angle. In the figures, however, by which these principles have been illustrated, we have employed the ordinary mode of representation. To the imperfections of this mode of representation we have already adverted, and it remains to be shown how these principles are practically applied in the constructions of problems embracing the three dimensions of space. In this really consists descriptive geometry.

In the ordinary mode of representing geometrical magnitudes of three dimensions, and likewise in ordinary drawing, the eye viewing any object is supposed to be fixed, and the several points in the object are referred to an ideal plane between it and the eye, by lines drawn from the eye through these points, and intersecting this plane, which is that of the picture: the intersections of these lines with the plane of the picture, are the representations of the respective points of the object. This is the principle of Perspective Drawing.

Considering the orthographic projection of points as a mode of viewing them, the eye must no longer be considered to be fixed, but to be successively transferred over every point, so as to be in that perpendicular to the plane of projection which passes through the point, and we may consider the plane to which the points are referred to be beyond the object. If any series of points were thus orthographically viewed with reference to two planes, we should have an orthographic picture of them on each plane, and these two pictures properly combined, that is, in the two planes of projections, would, according to the foregoing principles, give these points in their true positions in space. It is evident that we may suppose

each of these pictures to be made separately on paper. If the line of intersection of the planes of projection be made common to the two pictures, so that one is immediately above the other on the paper, they will be combined in one drawing; and in order to have these orthographic pictures in their true position, it is only necessary that the paper should coincide with one of the planes of projection, that, for instance, which is represented below their intersection; and that the paper above this intersection should then revolve about this line until it coincides with the other plane of projection.

Thus, let ab (fig. 17) be the picture of a line thus orthographically represented on a plane here indicated by $xyzw$; and $a'b'$ the picture of the same line orthographically represented on another plane indicated by $xyuv$; the two pictures having been separately made, but being here so combined that the line xy, representing the intersection of the planes of projection, is common to both. If the plane of the paper coincide with the original position of one of the co-ordinate planes xyz, the line xy coinciding with xy, the actual intersection of these planes, then if the paper above xy revolve about that line until the plane xyu coincides with the other of these planes, the projections ab, $a'b'$ will be in their true positions, and the true position in space of the line of which they are the projections will be given by the intersection of planes passing through ab, $a'b'$ and perpendicular to the respective planes of projection.

28. This manner of representing the two orthographic projections is in fact the same as the following. That all the constructions may be connected in a single drawing, one of the planes, xyu (fig. 18), for example, with the several lines and points on it, is made to turn about its intersection xy with the other plane xyz, until it coincides with the plane of xyz, in xyu'. In this manner, the lines traced upon the plane xyz, not having changed, are in their proper position; but to have a correct idea of the true position of those in the other plane, we must imagine this plane replaced in its proper position. As to points out of these planes, they ought not to appear in the figures.

29. After the turning down of one of the planes of projection, the projections of the same point are connected in a manner which it is very important that we should notice. Let a, a' be these projections before the turning down of the plane xyu. Then, since the perpendiculars drawn from a, a' on xy fall on the same point m (19); and during the turning of xyu about xy, the straight line $a'm$ continues perpendicular to xy, when the plane xyu coincides with the plane of xyz in xyu', the line ma' becomes the prolongation ma'' of ma.

Hence this theorem, of almost continual use, that *the projections of a point upon the two planes are in the same perpendicular to the intersection of the planes.*

30. Although we have referred to the horizontal and vertical

planes as those usually employed in projections, we have hitherto made no hypothesis with respect to the inclination of the planes of projection. To render the constructions more simple, we shall now take these planes perpendicular to each other. The established custom is to consider one of the planes horizontal, although it may have any position whatever, and the other as vertical.

31. The intersection of the two planes is called the *Line of Level.*

32. The projections of any points, lines, or figures in space, on the horizontal plane, is called the *Plan*; and their projections on the vertical plane, the *Elevation* of these points, lines or figures.

33. A line or a plane is said to be horizontal or vertical according as it is parallel or perpendicular to the horizon.

34. Projections and traces are called horizontal or vertical according as they are in the horizontal or vertical plane.

35. Several consequences arise from the planes of projection being perpendicular to each other.

1. *If a point or a line be in one of the two planes, its projection upon the other plane will be in the line of level.*

For the perpendiculars which determine this projection are wholly in the first plane (Prop. 33).

2. *If a line be in a plane parallel to one of the two planes, its projection upon the other plane will be a straight line parallel to the line of level.*

For example, let a line be in a horizontal plane; then the perpendiculars which determine its vertical projection will be wholly in this horizontal plane (Prop. 33); and its vertical projection will be the intersection of this plane with the vertical plane, which is a straight line parallel to the line of level (Prop. 27).

3. *If a plane be perpendicular to one of the planes of projection, its trace upon the other plane will be perpendicular to the line of level.*

For example, the vertical trace of a plane perpendicular to the horizon, that is, of a vertical plane, is perpendicular to the line of level (Prop. 34).

4. *The perpendiculars drawn on the line of level, from the projections of a point, are equal to the distances of this point from the two planes of projection.*

For a and a' (fig. 19) being the projections of the point A upon the planes xyz, xyu, when the planes are perpendicular, the angle ama' is a right angle, and $\mathrm{A}ama'$ is a parallelogram. Therefore $a'm=\mathrm{A}a$ and $am=\mathrm{A}a'$.

When the vertical plane xyu is turned down upon the horizontal plane xyu', the line ama'' drawn through the projections a, a'' of the same point A is perpendicular to the line of level xy; and it here appears further that the part ma'' gives immediately the elevation of this point above the horizontal plane, and that ma is its distance from the vertical plane.

36. When two planes intersect each other, they form, produced if necessary, four dihedral angles. Hitherto we have only considered the points projected to be posited in one of these angles; but it is evident that different points may be in any of the four angles, and it is important that we should examine how the representations of the projections of a point will be affected by the circumstance of its being posited in one or another of these angles.

Let *qrwz*, *vtsu* (fig. 20) be two planes perpendicular to each other, intersecting in *xy*, *qrwz* being horizontal and *vtsu* vertical, and forming the four dihedral angles *uxyz*, *uxyq*, *sxyq*, *sxyz*. Let the plane $AA_{/}A_{//}A_{///}$ be at right angles to the intersection *xy* of the planes, and therefore at right angles to each of the planes, intersecting them in $aa_{/}$, $\alpha\alpha_{//}$. Let αA, $\alpha A_{/}$, $\alpha_{//}A_{//}$, $\alpha_{//}A_{///}$ be perpendicular to the plane *vtsu*, and equal to each other; and aA, $aA_{///}$, $a_{/}A_{/}$, $a_{/}A_{//}$ perpendicular to the plane *qrwz*, and also equal to each other. Then A, $A_{/}$, $A_{//}$, $A_{///}$ will be corresponding positions of a point in each of the four dihedral angles; and the horizontal and vertical projections of A, $A_{/}$, $A_{//}$, $A_{///}$ will be respectively a and α, $a_{/}$ and α, $a_{/}$ and $\alpha_{//}$, a and $\alpha_{//}$. When the vertical plane *vtsu* is turned round upon *xy*, the upper part *xu down* upon *xq*, and the lower part *xs up* upon *xz*, the horizontal and vertical projections of A, $A_{/}$, $A_{//}$, $A_{///}$ will be represented respectively by a and a', $a_{/}$ and a', $a_{/}$ and $a'_{//}$, a and $a'_{//}$. Representing these projections in their proper positions with respect to the intersection *xy*, a and a' (fig. 20. 1) are the horizontal and vertical projections of A; $a_{/}$ and a' (fig. 20. 2) the projections of $A_{/}$; $a_{/}$ and $a'_{//}$ (fig. 20. 3) the projections of $A_{//}$; a and $a'_{//}$ (fig. 20. 4) the projections of $A_{///}$. To have them in their proper positions with respect to the horizontal and vertical planes, it is necessary that in fig. 20. 1, and fig. 20. 2, the plane *xyu'v'* should turn *up* on *xy*, as an axis, until it is perpendicular to the plane *xyz*; and that in fig. 20. 3, and in fig. 20. 4, the plane *xys't'* should turn *down* on *xy*, as an axis, until it is perpendicular to the plane *xyq*.

From this it follows, in the representation of these projections, that, taking any line *xy* on the paper to represent the line of level, the horizontal projections of all points which are in *front* of the vertical plane of projection being represented *below* the line of level *xy*, those of all points which are *behind* the vertical plane will be represented above that line; and the vertical projections of all points *above* the horizontal plane being represented above the line of level *xy*, those of all points *below* the horizontal plane will be represented below that line.

37. We now proceed to the solutions of the principal problems which may be proposed on the straight line and the plane.

In the figures which contain all the constructions of a problem, the data and the results will always be represented by full lines, and the lines in the construction by dotted or broken lines, which may be varied when necessary.

To obviate the confusion that might arise when points below the line of level represent vertical projections, or when points above that line represent horizontal ones, the following notation is adopted.

The capitals A, B, C, &c. denote points in space, and rarely appear in the figures.

The small letters a, b, c . . without accents belong to the horizontal plane.

When the letters are accentuated as a', b', c' . . . they belong to the vertical plane.

The line of level is always denoted by xy; and most commonly the Greek letters α, β, γ . . . indicate points in this line.

Different abbreviations sanctioned by custom are also employed.

The point [a, a'] is that of which the projections are a, a'.

The line [ab, $a'b'$] is that of which the projections are ab, $a'b'$.

The plane $a\alpha a'$ is that of which the traces are αa, $\alpha a'$.

When it is said that a point, a line, or a plane is given, it is to be understood that the projections of the point, those of the line, or the traces of the plane, are known. In the same manner, when it is proposed to determine a point, a line, or a plane, the projections of the point or of the line, or the traces of the plane, will be what it is required to find.

In the reasoning on which the construction of a problem is founded, it is always to be understood, that the points and lines referred to are considered to be in their true positions in space, with reference to the planes of projections, likewise in their true positions; that in the different operations of the construction, the vertical plane is supposed to have turned about the line of level until it coincides with the horizontal plane on that of the paper; and that to have a correct idea of the result of the construction, it is necessary mentally to restore the vertical plane, with the several points and lines on it, to its true position at right angles to the horizontal plane, so that the points and lines whose projections are represented on the two planes may, by means of these projections, be viewed in their true positions in space. The student, bearing this carefully in mind, is therefore recommended, after having performed all the operations of the construction, to fold the paper on the line of level, and having placed the portion representing the vertical plane at right angles to that representing the horizontal plane, to endeavour to obtain a clear notion of the positions in space, of the points or lines which were to be determined, by considering what must be the position of the intersection of their projecting lines or planes.

PROBLEMS.

INTERSECTIONS OF STRAIGHT LINES AND PLANES.

PROBLEM I.

A straight line being given, that is, *its projections being known; to find its traces,* that is, *the points where the straight line meets the planes of projection.*

The straight line in space is the intersection of its two projecting planes (20). But the point of meeting of the horizontal traces of these two planes is common to both planes, and is therefore a point in the straight line. As it is in the horizontal plane, it is therefore the horizontal trace of the straight line.

It appears in the same manner that the intersection of the vertical traces of the two projecting planes is the vertical trace of the straight line.

Let ab, $a'b'$ (fig. I. 1) be the two given projections of the straight line, which meet the line of level xy in a and b'. The plane perpendicular to the plane $yb'a'$ and passing through $a'b'$ is one of the projecting planes of the given straight line (11); and the horizontal trace of this plane is the line $b'h$, perpendicular to xy (35. 3). The horizontal trace of the vertical plane passing through ab, the other projecting plane, is ab. As the intersection h of these horizontal traces is a point in the line, h is the horizontal trace of the line.

Similarly, the vertical trace of the vertical plane passing through ab is av perpendicular to xy, and the vertical trace of the other projecting plane is $a'b'$. As the intersection v of these vertical traces is a point in the line, v is the vertical trace of the line.

Hence we have the following general rule. To determine the horizontal trace of a given straight line, produce its vertical projection to intersect the line of level, and from the point of intersection draw a perpendicular to this line, meeting the horizontal projection: the point of intersection will be the horizontal trace of the given line.

Similarly, to determine the vertical trace of a given straight line, produce its horizontal projection to intersect the line of level, and from the point of intersection draw a perpendicular to this line, meeting the vertical projection: the point of intersection will be the vertical trace of the given line.

Remarks.—In changing the projections ab, $a'b'$, the traces h, v may take an infinity of different positions. We shall notice four principal cases.

In fig. I. 1, the horizontal trace h is in front of the line of level, and the vertical trace v is above it.

In fig. I. 2, the horizontal trace h is still in front of the line of level; but the vertical trace v is in the lower part of the vertical plane.

In fig. I. 3, the reverse is the case: the horizontal trace h is behind the line of level, and the vertical trace v is above it.

In fig. I. 4, the horizontal trace h is behind the line of level, and the vertical trace v is below it.

By conceiving the portion of the figure above xy which represents the vertical projection of the given line, to revolve about xy, until it is perpendicular to the plane of the paper, and then in imagination supplying the projecting plane, a clear idea of the position in space of the given line, in the foregoing cases, will be obtained.

Problem II.

The traces of a straight line being given, to determine its horizontal and vertical projections.

Let h and v (fig. II. 1) be the given traces of a line. Then the plane xvy being in its original position, perpendicular to the plane xhy, the line joining h and v will be the line of which these points are the traces; and if va be drawn perpendicular to xy, and ha be joined, the plane hav will be its horizontal projecting plane, and therefore ha its horizontal projection. Similarly, if hb be drawn perpendicular to xy and vb' be joined, the plane $hb'v$ will be the vertical projecting plane of the line, and $b'v$ its vertical projection.

Problem III.

To find the intersection of two given planes; that is, *the traces of two planes being given, to find the projections of their intersection.*

If the traces of the two planes intersect in two points, these points being the traces of the line of intersection of the two planes, the projections of this line are immediately found by the last problem.

Let pa, pa' (fig. III.) be the horizontal and vertical traces of one of the given planes, and qb, qb' those of the other, m being the intersection of the horizontal, and n' that of the vertical traces. Then m is the horizontal trace of the line of intersection of the two planes; and n' is the vertical trace of that line. Therefore, drawing mm' perpendicular to xy, m' is the vertical projection of m (35. 1), and drawing nn' perpendicular to xy, n is the horizontal projection of n' (35. 1); joining therefore mn, $m'n'$, mn is the horizontal projection of the intersection of the two planes, and $m'n'$ is its vertical projection (Prob. II.).

Particular cases.—1. When one of the given planes, bqb', has one of its traces perpendicular to the line of level, the horizontal

trace bq (fig. III. 1), for example; then this plane is perpendicular to the vertical plane (Prop. 32), and therefore its vertical trace will be the vertical projection of its intersection with the other plane. In this case, the lines mm', $m'n'$ (fig. III.) themselves become the horizontal and vertical traces mq, qn', of the planes: in other respects the construction is the same as before.

2. Suppose that on one of the planes of projection, the traces of the two given planes are parallel to each other. For example, let the horizontal traces pa, qb (fig. III. 2) be parallel. In this case the intersection of the two planes will be parallel to pa and qb (Prop. 19. Cor.); and therefore its horizontal projection will also be parallel to them (Prop. 23). $n'n$ being therefore drawn, as before, perpendicular to xy, mn, drawn parallel to pa or qb, is the horizontal projection of the intersection of the two planes. Since the intersection is parallel to the horizontal plane (Prop. 21), its vertical projection is the line $n'm'$ drawn parallel to xy (35. 2).

3. Let the two given planes cut the line of level at the same point. In this case, the general construction fails, but we may cut the planes by any other plane, and find, as above, the projections of the line of intersection of each of the given planes with this auxiliary plane. The points where these projections meet each other will be the projections of a point common to both the given planes; and as the point where these planes meet the line of level is also common to them, the projections of their intersection are completely determined.

Very commonly the auxiliary plane is taken perpendicular to the line of level. It is then considered as a new plane of projection on which the traces of the given planes being found, the construction is by this means reduced to that of the general case.

Let apa', bpb' (fig. III. 3) be the given planes, cutting the line of level xy in the same point p, their traces being ap, $a'p$ and bp, $b'p$. Let an auxiliary plane be conceived to be drawn perpendicular to xy, its traces on the co-ordinate planes being $\beta\alpha$, $\beta\alpha'$, which meet the traces of the plane apa' in a and a'; then if we conceive the vertical plane of projection to be in its true position, perpendicular to the horizontal plane, the line joining a and a' in the auxiliary plane will be the trace of the plane apa' upon this plane. Similarly, the line joining b and b', in the auxiliary plane, will be the trace of the plane bpb' on this plane. Let us now conceive that this vertical plane is turned about $\beta\alpha$, down upon the horizontal plane, so that the line $\beta\alpha'$ is upon βy. The point a will not be changed, but a' will be brought down to the point a'', so that $\beta a'' = \beta a'$; and consequently the straight line joining a and a' in space will be brought down in aa''.

In the same manner, the trace bb'', of the other given plane, on the auxiliary plane, is found; the point d where it cuts aa'' is com-

mon to the two planes; and we have only now to determine the projections of this point upon the primitive co-ordinate planes.

Now if dm be drawn perpendicular to $\beta\alpha$, the horizontal projection of d will be m; and if dd' be drawn perpendicular to βy, and $\beta d'$ be brought back to $\beta m'$ on $\beta\alpha'$, the vertical projection of d will be m'*. The projections of the required intersection of the planes will therefore be pm, pm'.

4. Let one of the given planes be parallel to the line of level.

In this case, the traces bq, $b'q'$ (fig. III. 4) of one of the planes are parallel to the line of level xy (Prop. 23). m being the intersection of the horizontal traces of the two planes, and n' that of their vertical traces, the horizontal and vertical projections, mn and $m'n'$, of the intersection of the planes will be found as before, by drawing mm' and $n'n$ perpendicular to xy.

5. Let both the given planes be parallel to the line of level.

In this case, the traces of the given planes are parallel to the line of level (Prop. 23); their intersection is so likewise (Prop. 23, Cor. 2); and the general construction again fails. But this defect is obviated by means of an auxiliary plane, as in the 3rd case.

Let ap, $a'p'$ (fig. III. 5) be the horizontal and vertical traces of one of the given planes, and bq, $b'q'$ those of the other, all parallel to the line of level xy. Let $\beta\alpha$, $\beta\alpha'$ be the horizontal and vertical traces of an auxiliary plane, which is here taken to be oblique to xy, intersecting ap, bq in c, e, and $a'p'$, $b'q'$ in d', f'; c and d' will be the horizontal and vertical projections of the intersection of the plane ap, $a'p'$ with the auxiliary plane, and e and f' those of the intersection of the plane bq, $b'q'$ with the same plane.

Draw cc' and $d'd$ perpendicular to xy, and join cd, $d'c'$: cd is the horizontal projection of the intersection of the plane ap, $a'p'$ with the auxiliary plane $\alpha\beta\alpha'$, and $d'c'$ is the vertical projection of that intersection (Case 4). Again, drawing ee' and $f'f$ perpendicular to xy, and joining ef, $f'e'$, ef and $f'e'$ are the horizontal and vertical projections of the intersection of the plane bq, $b'q'$ with the plane $\alpha\beta\alpha'$. Therefore cd and ef being the horizontal projections of the intersections of the two given planes with the auxiliary plane, m, the point where these lines intersect each other, will be the horizontal projection of the point of intersection of the two given planes on the auxiliary plane; and $d'c'$ and $f'e'$ being the vertical projections of those intersections, m', the point where they intersect, will be the

* The operation by which the vertical projection of the intersection of the traces of the given planes on the auxiliary vertical plane is represented in the drawing, is simply this: after the determination of the point d', the line $\beta\alpha'$ is supposed to be in its correct position perpendicular to the plane $x\beta\alpha$, the line $\beta d'$ is then turned up on $\beta\alpha'$, to determine the point m', and then the line $\beta\alpha$, with this point m' determined on it, is again brought down upon the paper, supposed throughout to be horizontal, as $\beta m'a'b'$.

vertical projection of the same point of intersection. Since then m and m' are the projections of a point common to the two planes, and that the intersection of the planes is parallel to xy (Prop. 23, Cor. 2), mn, $m'n'$ parallel to xy are the horizontal and vertical projections of that intersection.

Problem IV.

To find the intersection of a given line with a given plane.

The method of solution consists in drawing any plane to pass through the given line; finding the line of intersection of this plane with the given plane; and determining the intersection of the two straight lines.

The construction is rendered very simple by taking the auxiliary plane perpendicular to one of the planes of projection. Let αa, $\alpha a'$ (fig. IV.) be the traces of the given plane, and bc, $b'c'$ the projections of the given straight line.

Let us first take for the auxiliary plane the vertical plane which contains the given line, that is, the horizontal projecting plane of this line. Its horizontal trace will be bc, and its vertical trace mm' perpendicular to xy. Let these traces meet αa and $\alpha a'$ in d and m'; then drawing dd' perpendicular to xy, and joining $m'd'$, this will be the vertical projection of the intersection of the two planes (Prob. III. 1). As the given line must intersect the given plane somewhere in the intersection of the given plane and the auxiliary plane which contains that line, the vertical projection of the required point of intersection must be somewhere in $m'd'$. It must likewise be somewhere in $b'n'$, the vertical projection of the given line; it is therefore in o', the intersection of $m'd'$ and $b'n'$.

Taking, in like manner, for the auxiliary plane the vertical projecting plane of the given line, its traces will be $n'b'$ and $n'n$, perpendicular to xy, cutting $\alpha a'$ and αa in e' and n. Drawing $e'e$ perpendicular to xy, and joining ne, these are the projections of the intersection of the auxiliary plane with the given plane. The horizontal projection of the required point of intersection must be in the line ne, and it must also be in the line bm; it must therefore be their intersection o.

When the points o and o' are correctly determined in the figure, the line oo' is perpendicular to xy.

We may remark one particular case: when the given line is perpendicular to one of the planes of projection; for example, to the horizontal plane. Its projection on this plane is a single point o (fig. IV. 1), which suffices to determine it; and its vertical projection is $o''o'$ perpendicular to xy. Here the auxiliary plane $d'odd'$ must necessarily be vertical, and its vertical trace dd' perpendicular to xy; and its horizontal trace is only restricted to the condition of

passing through the point o. $c\,c'$ being drawn perpendicular to xy, $d'c'$ is the vertical projection of the intersection of the two planes; and therefore o', the point where it cuts the vertical projection of the given line, is the vertical projection of the required point of intersection.

The construction may be modified by taking the horizontal trace of the auxiliary plane parallel to αa the horizontal trace of the given plane, as oe; or parallel to the line of level, as of. In the first case, the intersection of the two planes will also be parallel to αa, and its vertical projection $e'o'$ will be parallel to xy (Prob. III. 2). In the second case, the intersection of the planes will be parallel to $\alpha a'$ (Prop. 27), and consequently its vertical projection $f'o'$ will be so likewise (Prop. 27).

In the particular case which we have just examined, we may consider the point o as the horizontal projection of a point in the given plane $a\alpha a'$, and then it is evident that this last point must be in the vertical line drawn from o. We thus see that the preceding construction will resolve this problem: *one of the projections of a point in a plane being given, to find the other projection.*

Problem V.

To determine the point of intersection of three given planes.

The intersection of any two of the planes is a straight line, and the intersections of the third plane with each of these planes will meet in a point of their common intersection. The intersections, therefore, of the planes, taken two and two, determine three lines which pass through the required point. The projections of these intersections are found by Prob. III.; and when the constructions are correct, the three horizontal projections meet in a point which is the horizontal projection of the point of intersection of the three planes; the three vertical projections likewise meet in a point which is the vertical projection of this point; and, moreover, the straight line joining these two points, which are the projections of the same point in space, is perpendicular to the line of level.

Let αa and $\alpha e'$, βb and $\beta d'$, γc and $\gamma f'$ (fig. V.) be the traces of the three planes; then, constructing as in the figure according to Prob. III., the horizontal projections of the intersections of the three planes, taken two and two, are ad, bf, ce; their vertical projections are $a'd'$, $b'f'$, $c'e'$; and the projections of the required point are o and o'.

Problem VI.

The projections of two points being given, to find the projections and the true length of the line joining these points.

The projections of the line will be the lines joining the projections of its extremities in each plane (10).

Let a, a' and b, b' (fig. VI.) be the projections of the two points A, B. It is evident that, joining ab, $a'b'$, these are the projections of the straight line joining the points A and B in space.

To find the true length of the straight line AB.

Since $a'a$ and $b'b$ are perpendicular to the line of level xy (29), $a'r$, $b's$ are equal to the perpendiculars to the horizontal plane drawn from a and b to the points A and B in space. These two vertical lines being in the same plane and perpendicular to ab, it is evident that, if through B we conceive a line parallel to ab and terminated by the other vertical through A, a right-angled triangle will be formed, having for its base this parallel equal to ab; for its altitude the difference of the two verticals, that is, of $a'r$ and $b's$; and for its hypothenuse the required distance AB. If therefore through b', mn be drawn parallel to xy, and equal to ab; and $a'n$ be joined: $a'n$ will be the required distance. This is in fact nothing more than conceiving the vertical plane passing through AB, ab to be applied to the plane xra' so that a coincides with r, and ab with rp: the point A will then coincide with a', and B with n. We may readily conceive this superposition to take place thus: the plane xra' being in its true position, that is, perpendicular to the horizontal plane xra, imagine the vertical plane passing through AB, ab to revolve about Aa, until it is parallel to the vertical plane xra', ab being in the position ac; and then conceive this plane to move parallel to itself until it coincides with the vertical plane $pra'n$.

If the vertical plane passing through ab be conceived to be turned about ab, down upon the horizontal plane, then drawing bg and af perpendicular to ba, and respectively equal to $b's$ and $a'r$, the straight line AB will coincide and be equal to gf.

Problem VII.

Through a given point, to draw a straight line parallel to a given straight line; that is, having given the projections of a point and of a straight line, to find those of a parallel to the straight line, drawn through the point.

Since the line passing through the given point is to be parallel to the given line, the projecting planes of the required line, and those of the given line, will be respectively parallel (Prop. 35); and consequently the intersections of the projecting planes with the planes of projection, that is, the projections of the lines, will be re-

spectively parallel (Prop. 27). Let ab, $a'b'$ (fig. VII.) be the projections of the given straight line, and c, c' those of the given point. If therefore through c and c', cd and $c'd'$ be drawn respectively parallel to ab and $a'b'$, cd and $c'd'$ will be the projections of the line passing through the point [c, c'], parallel to the line [ab, $a'b'$].

Problem VIII.

One projection of a straight line in a given plane, being given, to find the other projection of that line.

Let $a\alpha a'$ (fig. VIII.) be the given plane, and suppose the horizontal projection of a line in it is given, and that this projection is bc. If through bc we conceive a vertical plane to pass, its vertical trace will be cc' perpendicular to xy; and its intersection with the plane $a\alpha a'$ will be the line of which bc is the horizontal projection. We have, therefore, only to find the vertical projection of this intersection by drawing $c'b'$ to the foot of the perpendicular from b on xy (Prob. III. 1).

Problem IX.

Through a given point, to draw a plane parallel to a given plane.

Let [a, a'] (fig. IX.) be the given point, and $b\beta b'$ the given plane. Since the required plane is to be parallel to the plane $b\beta b'$, its traces on the planes of projection will be parallel to βb and $\beta b'$ (Prop. 27); and if a horizontal plane be supposed to pass through the point [a, a'], its intersection with the required plane will be parallel to βb (Prop. 27). The horizontal projection of this intersection will therefore be parallel to βb, and pass through a; and its vertical projection will be parallel to xy, and pass through a' (35—2). Drawing, therefore, through a, ad parallel to $b\beta$; from d drawing dd' perpendicular to xy; and through a', $a'd'$ parallel to xy; ad and $a'd'$ will be the horizontal and vertical projections of this intersection, and d' will be the point where this intersection cuts the vertical plane; that is, d' is a point in the vertical trace of the required plane. Drawing, therefore, through d', $\gamma c'$ parallel to $\beta b'$, $\gamma c'$ will be the vertical trace of the required plane.

In the same manner drawing through a', $a'e'$ parallel to $b'\beta'$; from e', $e'e$ perpendicular to xy; and through a, ae parallel to xy; e will be a point in the horizontal trace of the required plane. Drawing through e, γec parallel to βb, γc is the horizontal trace of that plane.

Otherwise. If a straight line be drawn in the given plane, a straight line parallel to it and passing through the given point will be wholly in the plane required (Prop. 23. Cor. 1). The traces of this last line are therefore points in the traces of the plane required; and as these

traces must be parallel to those of the given plane (Prop. 27), they will be found by drawing, through the traces of this line, parallels to the traces of the given plane. Let a, a' (fig. IX. 1) be the projections of the given point, and βb, $\beta b'$ those of the given plane. In βb, $\beta b'$ take any points m, n' as the traces of a straight line in the given plane: draw mm' $n'n$ perpendicular to xy; and join mn, $m'n'$, these are the projections of the straight line whose traces are m, n' (Prob. II.). Through a, a', parallel to nm, $m'n'$, draw qp, $p'q'$, they are the projections of a line passing through the point [a, a'] parallel to the line [mn, $m'n'$] (Prob. VII.); and drawing $p'p$, qq' perpendicular to xy, p, q' are the traces of this line (Prob. I.). Through p and q' draw $p\gamma$, $q'\gamma$ parallel to $b\beta$, $b'\beta$: $p\gamma$, $q'\gamma$ are the traces of a plane parallel to the plane $b\beta b'$ (Prop. 27); and it passes through the given point [a, a'], because it passes through the line [pq, $p'q'$] in which that point is.

In both constructions, if they be correct, the straight lines drawn parallel to βb, $\beta b'$ will meet xy in the same point γ.

Problem X.

To draw a plane to pass through three given points; that is, the projections of three points being given, to determine the traces of the plane passing through these points.

If the three points be in a straight line, any plane whose traces pass through the traces of this line will pass through the three points.

When the three points are not in a straight line, the straight lines joining them two and two form a triangle, and the traces of its three sides will give three points in each of the traces of the plane passing through the three points. To resolve the problem, we have, therefore, only to find by Problem I. the traces of the three straight lines joining the points whose projections are given: if all the constructions be correct, the three points thus determined in each plane of projection will be in a straight line; and the straight lines drawn through these points will be the traces of the required plane.

Let a, b, c (fig. X.) be the horizontal, and a' b' c' the vertical projections of the three given points: then constructing as in the figure according to Problem I., h, v are the horizontal and vertical traces of the side [ab, $a'b'$]; h', v' are those of the side [ac, $a'c'$]; and h'', v'' those of the side [bc, $b'c'$]. Consequently, the horizontal trace of the required plane is the straight line passing through the points h, h', h''; the vertical trace is the straight line passing through v, v', v''; and these two traces must meet the line xy in the same point t.

In the particular case, where one of the lines joining two of the

points is parallel to one of the planes of projection, this line will give no trace on the plane to which it is parallel; but still the trace of the required plane, on this plane, will be determined by the traces of the other two lines, when neither is parallel to the same plane. And besides this, the trace of the required plane on the plane to which the line is parallel must be parallel to the projection of the line on this plane (Props. 23, 19).

If two of the lines joining two of the points to the third point be parallel to one of the planes of projection, the third line must also be parallel to the same plane, and the plane passing through the three points likewise parallel to it.

Problem XI.

To draw a plane to pass through a given point and through a given straight line.

This is only another form of the last problem, but in this form a very simple construction presents itself. If through the projections of the given point, straight lines parallel to the projections of the given line be drawn, these will be the projections of a straight line parallel to the given line and passing through the given point (Prob. VII.); and which therefore is in the required plane. The traces of these two lines will be points in the traces of the required plane.

Let a, a' (fig. XI.) be the projections of the given point, and bc, $b'c'$ those of the given straight line. Through a, draw de parallel to bc, and through a', draw $d'e'$ parallel to $b'c'$; de and $d'e'$ will be the projections of a line passing through the point [a, a'] and parallel to the line [bc, $b'c'$] (Prob. VII.). Through c' and e', draw $c'c$ and $e'e$ perpendicular to xy, and cutting bc and de in c and e; and through b and d, draw bb' and dd', cutting $c'b'$ and $e'd'$ in b' and d'. c and e are the horizontal traces, and b' and d' the vertical traces of the parallel lines [bc, $b'c'$] and [de, $d'e'$] (Prob. I.). Drawing therefore ect, $d'b't$, these are the required traces of the plane in which these parallels are.

When the construction is correct, ect and $d'b't$ will meet xy in the same point t.

Problem XII.

To draw a plane to pass through a given straight line, and to be parallel to another given straight line, not in the same plane with the first.

If through a point in the first line, a straight line be drawn parallel to the second, the plane passing through the former two lines will be parallel to the latter line (Prop. 21), and will, therefore, be the plane required.

Let [ab, $a'b'$] (fig. XII.) be the line through which the plane is to pass, and [cd, $c'd'$] that to which it is to be parallel. In [ab, $a'b'$] take any point [f, f']. Through f draw ge parallel to cd, and through f' draw $g'e'$ parallel to $c'd'$. From the points a' and e', draw $a'a$ and $e'e$ perpendiculars to xy, meeting ba and ge in a and e; and from b and g, draw bb' and gg' also perpendicular to xy, meeting $a'b'$ and $e'g'$ in b' and g': a and e are the horizontal traces, and b' and g' the vertical traces of the lines [ab, $a'b'$] and [eg, $e'g'$]. Drawing therefore eat and bgt, these are the traces of the plane required.

When the construction is correct, eat and $b'g't$ meet xy in the same point t.

Problem XIII.

Through a given point, to draw a plane parallel to two given straight lines.

If through the given point, straight lines be drawn parallel to the given straight lines, they will determine a plane (Prop. 2), and this plane will be parallel to each of the given straight lines (Prop. 21). The traces of the parallels to the given lines will, therefore, be points in the traces of the plane required, and will consequently determine the latter traces.

Let a, a' (fig. XIII.) be the projections of the given point, and bc, $b'c'$ and de, $d'e'$ those of the given straight lines. Through a, draw fg and hi respectively parallel to bc and de, and through a', $f'g'$ and $h'i'$, respectively parallel to $b'c'$ and $d'e'$. Then constructing as in the figure, according to Problem I., f and i are the horizontal traces, and h' and g' the vertical traces of straight lines in a plane parallel to the lines [bc, $b'c'$] and [de, $d'e'$], and passing through the point [a, a']. Drawing, therefore, ift and $g'h't$, these are the traces of that plane.

When the constructions are correct, the lines if and $g'h'$ meet xy in the same point t.

Problem XIV.

To draw a straight line that shall pass through a given point and meet two given straight lines.

It is evident that the given point and either of the given lines will determine a plane, in which all lines must be that pass through the given point and meet that line (Prop. 2).

If, therefore, through the given point and each of the given lines planes be drawn, the line required must be in each of these planes, and must therefore be their intersection which passes through the given point. If, however, either of the given lines be parallel to the

intersection of the two planes, no line can be drawn through the given point to meet both the given lines.

In order to determine the straight line required, we have therefore to determine by Problem XI., planes passing through the given point and each of the given lines, and then determine the intersection of these planes by Problem III.

Let a, a' (fig. XIV.) be the projections of the given point, bc, $b'c'$ and de, $d'e'$ those of the given straight lines. Constructing as in the figure, according to Problem XI., $b\alpha$ and $\alpha c'$ are the traces of the plane passing through the point [a, a'] and the line [$bc, b'c'$]; and $d\beta$ and $\beta e'$ are those of the plane passing through the point [a, a'] and the line [$de, d'e'$]. And again, according to Problem III., ik and $i'k'$ are the projections of the intersection of the two planes, that is, of the line required.

Three verifications of the constructions are to be noted.

1. The intersection of the two planes must pass through the given point; and therefore its projections must pass through those of the given point.

2. It must meet the line [$bc, b'c'$]; and consequently the intersections f, f' of its projections with those of this line must be in the same perpendicular to xy (29).

3. It must likewise meet the line [$de, d'e'$]; and therefore the intersections g, g', of its projections with those of this line must be in the same perpendicular to xy.

PERPENDICULAR STRAIGHT LINES AND PLANES.

Problem XV.

From a given point, to draw a straight line perpendicular to a given plane; and to determine the intersection of this perpendicular with the given plane, and also its true length.

Since the projecting planes of the required perpendicular are at right angles, both to the respective planes of projection and to the given plane (Prop. 32), they are at right angles to the respective traces of this plane (Prop. 34); and consequently the projections of the required perpendicular to the given plane are at right angles to the respective traces of this plane (Def. 3).

To find the projection of the required perpendicular, we have therefore only to draw, from the projections of the given point, perpendiculars to the respective traces of the given plane.

The intersection of the perpendicular with the given plane is found by Problem IV.; and its true length by Problem VI.

Let [a, a'] (fig. XV.) be the given point, and mam' the given plane. From a and a' draw ab and $a'b'$ respectively perpendicular to αm, $\alpha m'$: they are the projections of the required perpendicular.

Constructing as in the figure, according to Problem IV., b and b' are the projections of the intersection of the required perpendicular with the given plane.

And again, constructing according to Problem VI., drawing through b', ec parallel to xy and equal to ab, $a'c$ is the true length of the perpendicular.

Problem XVI.

Through a given point, to draw a plane and also a straight line, both perpendicular to a given straight line.

Since the projecting planes of the given straight line are perpendicular to the planes of projection, and also to the plane required (Prop. 32), they are perpendicular to the respective traces of that plane (Prop. 34. Cor. 2); and therefore these traces are perpendicular to the respective projections of the given line (Def. 3). If, therefore, we can determine a point in one of these traces, they may both be determined. Now if a horizontal plane pass through the given point, its intersection with the plane required will be parallel to the horizontal trace of this plane (Prop. 27), and will meet the vertical plane of projection in a point of the vertical trace of this plane: also the horizontal projection of this intersection will be parallel to the horizontal trace of the plane required (Prop. 19. Cor.), and therefore perpendicular to the horizontal projection of the given line; and its vertical projection will be parallel to the line of level (Prop. 27).

Let a, a' (fig. XVI.) be the projections of the given point, and bc, $b'c'$ those of the given line. From a, draw ad perpendicular to bc; from a', $a'd'$ parallel to xy; and from d, dd' perpendicular to xy. Then ad and $a'd'$ will be the projections of the intersection of a horizontal plane passing through the point [a, a'], with the plane through that point and perpendicular to the line [bc, $b'c'$]; and d' will be a point in the vertical trace of this plane. Through d' draw $m'\alpha$ perpendicular to $b'c'$, and from α draw αm perpendicular to bc. Then, from what has been stated, αm and $\alpha m'$ are the traces of the plane passing through the given point [a, a'] and perpendicular to the given line [bc, $b'c'$]. Constructing as in the figure, according to Problem IV., p and p' are the projections of the intersection of the line [bc, $b'c'$] with the plane $m\alpha m'$; and joining ap, $a'p'$ they are the projections of the line drawn from the point of intersection of the given line with the plane perpendicular to it, to the given point [a, a'] in that plane; that is, of the line perpendicular to the given line [bc, $b'c'$] (Def. 3) and passing through the given point [a, a'].

PROBLEM XVII.

Through a given straight line, to draw a plane perpendicular to a given plane.

If a perpendicular to the given plane, or to a plane parallel to it (Prop. 28), be drawn from a point in the given straight line, then it is evident that a plane passing through the foot of this perpendicular and the given straight line will be perpendicular to the given plane, and will therefore be the plane required. It may sometimes be more convenient to draw the plane perpendicular to a plane parallel to the given plane, than perpendicular to the given plane itself. We shall, therefore, give that construction, though the two constructions are essentially the same.

Let $[bc, b'c']$ (fig. XVII.) be the given line, of which the horizontal and vertical traces are c and b'; and let $a\alpha a'$ be the given plane. Through b' draw $d'\delta$ parallel to $a'\alpha$, and through δ, δd parallel to αa; the plane $d'\delta d$ is parallel to the plane $a'\alpha a$ (Prob. IX.), and passes through the vertical trace b' of the given line. From c and c' draw cm and $c'n'$ perpendicular to δd and $\delta d'$, and constructing as in the figure, according to Problem IV., q and q' are the projections of the intersection of the perpendicular drawn from the point $[c, c']$ upon the plane $d\delta d'$, with that plane. Again, drawing through q and q', pr and $p'r'$ respectively parallel to bc and $b'c'$; from p and r', pp' and $r'r$ perpendicular to xy; and joining rc, $p'b'$, and producing them to ϕ; $r\phi$ and $p'\phi$ are the traces of the plane passing through the point $[q, q']$ and the line $[bc, b'c']$ (Prob. XI.). Also, since the line $[cq, c'q']$ is perpendicular to the plane $d\delta d'$, and therefore to the parallel plane $a\alpha a'$ (Prop. 28), the plane $r\phi p'$, passing through the line $[cq, c'q']$ is perpendicular to the plane $d\delta d'$, and likewise to the plane $a\alpha a'$.

ANGLES OF STRAIGHT LINES AND PLANES.

PROBLEM XVIII.

To find the angles which a given straight line makes with the planes of projection.

Since the horizontal projecting plane of the given line is perpendicular to the horizontal plane of projection, its trace on the vertical plane is perpendicular to the horizontal projection of the line (Prop. 34 and Def. 3); and therefore (Def. 4) the angle which the line makes with the horizontal plane is the angle which it makes with its horizontal projection. Similarly, the angle which the line makes with the vertical plane is the angle which it makes with its vertical projection.

Let ab and $a'b'$ (fig. XVIII.) be the projections of the given line.

Then drawing $a'a$ and bb' perpendicular to xy, a and b' are the traces of this line on the horizontal and vertical planes of projection (Prob. I.). If now the vertical plane $xb'y$ be in its true position, perpendicular to the plane xay, the angle made by the given line with the horizontal plane is the angle between the line joining a and b' in space, and its horizontal projection ab (Def. 4). Let the plane abb' be conceived to turn about bb' until it coincides with the vertical plane; then the point a will be applied to xy, in m, making bm equal to ba; the line joining a and b' in space will be in the position $b'm$; and the angle bmb' will, therefore, be equal to the angle which the given line makes with the horizontal plane.

This angle may be brought down upon the horizontal plane by conceiving the plane abb' to turn about ab, the vertical line bb' being in this case brought down upon bn, the perpendicular to ab.

Similarly, the angle which the given line makes with the vertical plane is equal to the angle $a'pa$, by conceiving the plane $b'a'a$ to turn about $a'a$ on to the horizontal plane; or is equal to the angle $a'b'q$, by conceiving that plane to turn about $a'b'$ on to the vertical plane.

PROBLEM XIX.

Through a given point, to draw a straight line that shall make given angles with the planes of projection.

The angles which a line makes with the planes of projection are the angles which it makes with its projections in those planes (Prob. XVIII.). It is therefore these angles which it is required to make equal to the given angles.

If a straight line be drawn through any point making the given angles with the planes of projection, then a straight line parallel to this line, and passing through the given point, will also make the given angles with those planes, because the lines and their projections being respectively parallel, they will contain equal angles (Prop. 20), and this straight line will therefore be the one required.

Let $[m, m']$ (fig. XIX.) be the given point, and α, β the given angles. At any point B, in xy, make the angle $a'Ba$, in the vertical plane equal to the angle β, and draw $a'a$ perpendicular to xy. Then conceiving the plane $aa'B$ to turn about aa', the point B will describe a circle, and the straight line joining any point in this circle with the point a' will make an angle equal to β with the horizontal plane; and the length of this line will be $a'B$, being the hypothenuse of a right-angled triangle of which $a'a$ and ab or aB are the sides. Of such lines it is therefore required to find that which shall make with its vertical projection an angle equal to α. We have, therefore, to determine a right-angled triangle of which the hypothenuse is $a'B$, and the vertical angle at a' is equal to α. At a' make the angle

$Ba'B'$ equal to α, and upon $a'B$ as a diameter describe a semicircle, cutting $a'B'$ in B'; join BB': BaB', is the triangle required; and $a'B$ is the length of the vertical projection of the required line. Describing, therefore, a circle from a' as a centre to meet xy in b', $a'b'$ is the vertical projection of the required line; and drawing $b'b$ perpendicular to xy, joining ab, ab is its horizontal projection: that is, the line joining $a'b$ in space makes with its horizontal projection ab an angle equal to β, and with its vertical projection $a'b'$ an angle equal to α.

When the construction is correct, $b'b$ is equal to $B'B$.

Through m, m' draw np, $n'p'$ parallel to ba, $b'a'$; then these are the projections of the line passing through the point $[m, m']$ and making the angles β, α with the planes of projection.

Remarks.—1. Since a line makes with its projection on either plane a less angle than with any other line meeting it in that plane (Prop. 15), the line joining b and a' in space makes a less angle with $a'b'$ than with $a'a$; but the latter angle, together with that made by the same line with ab is equal to a right angle: consequently, the angles which the line joining b and a' in space makes with its projections ab, $a'b'$, do not together exceed a right angle. It follows from this that the given angles α and β together must not exceed a right angle.

2. The same limitation is manifest from the construction; for if the angles $a'Ba$ and $Ba'B'$ were together greater than a right angle, the angle $Ba'B'$ would be greater than $Ba'a$; and the circle described from the centre a', at the distance $a'B'$, would not meet aB.

3. As the circle $b'B'$, except in the case of $a'B'$ being equal to $a'a$, will meet Ba produced, there are two lines, on contrary sides and equally distant from $a'a$, which make angles α, β with the planes of projection, except in the case in which these angles are together equal to a right angle.

Problem XX.

Through a given point, to draw a straight line parallel to a given plane, and making, with the horizontal plane, a given angle, not greater than the inclination of the given plane to the horizontal plane.

It is evident that if through the given point a plane be drawn parallel to the given plane, all lines in the parallel plane will be parallel to the given plane; and consequently all lines parallel to the given plane, and passing through that point, will lie in the parallel plane. The problem is therefore reduced to finding in this parallel plane, the straight line which drawn through the given point shall make a given angle with the horizontal plane.

Let $[a, a']$ (fig. XX.) be the given point, $b\beta b'$ the given plane, and LMN the given angle: it is required to find the projections of

the straight line which, drawn through the point $[a, a]$, parallel to the plane $b\beta b'$, shall make with the horizontal plane an angle equal to LMN.

Through a, draw ad parallel to $b\beta$; from d, draw dd' perpendicular to xy; through a', draw $a'd'$ parallel to xy; through d', draw $\gamma c'$ parallel to βb; and through γ, draw γe parallel to βb: γe, $\gamma c'$ are the traces of the plane parallel to the plane $b\beta b'$, and passing through the point $[a, a']$ (Prob. IX.).

From any point L, in LM, draw LN at right angles to MN; and at the point d' draw $d'm$, making the angle $dd'm$ equal to the angle MLN, and therefore the angle $d'md$ equal to the given angle LMN. From the centre d, at the distance dm, describe a circle cutting γe in f; from f, draw ff' perpendicular to xy; and join df, $d'f'$. Through a and a', draw ce, $c'e'$ respectively parallel to df, $d'f'$, and meeting γf, $\gamma d'$ in e, c'; ce, $c'e'$ are the projections of the straight line passing through the point $[a, a']$, parallel to the line $[df, d'f']$ (Prob. VII.); e, c' are the traces of this line, because the point $[a, a']$ being in the plane $f\gamma d'$ the line $[ce, c'e']$ parallel to $[df, d'f']$ in that plane is in the same plane, and its traces are in the traces of that plane; and cc' and ee' are therefore perpendicular to xy (Prob. I.). The straight line $[ce, c'e']$ is that required.

Conceive the plane $xc'y$ to be in its original vertical position; then the angle $d'df$ being a right angle, the triangles $d'df$ and $d'dm$ are each right-angled at d, and they have df equal to dm, and $d'd$ common; therefore the angle at f, made with fd by the line joining $d'f$ in space, or by the line $[df, d'f']$, is equal to the angle $d'md$, that is, to LMN.

The angle which the line $[df, d'f]$ makes with the horizontal plane being the angle which it makes with its horizontal projection df, is therefore equal to the given angle LMN.

The projections of the lines $[ce, c'e']$, $[df, d'f']$ being parallel, and also the lines themselves, the angles made by these lines with their respective projections are equal; consequently the line $[ce, c'e']$ makes with the horizontal plane an angle equal to the given angle LMN: it passes through the given point $[a, a']$; and it is parallel to the given plane $b\beta b'$, because it is the plane $c'\gamma e$ which is parallel to the plane $b\beta b'$.

Remarks.—1. The angle formed with df, by the line joining d', f in space, may also be shown to be equal to the angle $d'md$, by conceiving the plane $d'df$ to turn about $d'd$ until it coincides with the vertical plane, when this angle will coincide with $d'md$.

2. When dm is such that the circle described at this distance from d as a centre touches γe, the line df being perpendicular to γe, the angle made with the horizontal plane by the line joining d', f in space is the angle of inclination of the plane $c'\gamma e$ to the horizontal plane, and no line can be drawn in the plane $c'\gamma e$, making a greater

angle with the horizontal plane than this angle of inclination (Prop. 15).

3. When the circle cuts γe, it cuts it in two points, and therefore there may be drawn two straight lines passing through the given point $[a', a]$, in the plane $c'\gamma e$, which make the required angle with the horizontal plane.

Problem XXI.

To determine the angle of inclination of a given plane to each of the planes of projection, and also the angle between its two traces.

If from any point in the line of level, a perpendicular to that line be drawn to meet the vertical trace of the given plane, and from the same point, a perpendicular be drawn on its horizontal trace, the line in space joining the intersections of these perpendiculars with the two traces will be perpendicular to the horizontal trace (Prop. 16); and therefore the angle between this line and the perpendicular on the horizontal trace, is the angle which the given plane makes with the horizontal plane (Def. 7).

Let qp, qr (fig. XXI.) be the traces of the given plane. From any point b in xy draw bc perpendicular to xy, and ba perpendicular to qp; the angle between the line joining a and c in space, and the line ba, is the angle which the plane pqr makes with the horizontal plane. In order to obtain this angle on the vertical plane, conceive the plane cba to turn about cb until it coincides with the former plane; when ba will coincide with bA, and cAb will be the angle between the two perpendiculars to the horizontal trace qp, or the angle which the given plane makes with the horizontal plane.

To refer the same angle to the horizontal plane, conceive the plane cba to turn about ab down upon the former plane, so that the line bc is brought in the position bC, perpendicular to ab: then baC will be the angle required.

Similarly, drawing be perpendicular to xy, and bd perpendicular to the vertical trace, the angle which the given plane makes with the vertical plane, will be found to be bDe, by conceiving the plane dbe to turn about be until it coincides with the horizontal plane in Dbe; or to be bdE, by conceiving the same plane to turn about db until it coincides with the vertical plane, the line be being brought in the position bE, perpendicular to db.

To find the angle between the two traces of the given plane.

Since the line joining a and c in space is perpendicular to qp, if the plane cqa be conceived to turn about qa, until it is brought down upon the horizontal plane, that line will be in the position ac' perpendicular to qa; the point c will be transferred to c', making qc' equal to qc; and the angle cqa, in the plane pqr between its two traces, will coincide and be equal to the angle $c'qa$.

Cor. From *b* draw *b*F perpendicular to *c*D, and *b*G perpendicular to *a*C; let the circle *c*CH cut D*e* in H; and join *b*H: the angle F*b*H is equal to the angle of inclination of the plane *pqr* to the horizontal plane.

The plane *qcy* being in its vertical position, the planes *ebd* and *abc* are the projecting planes of the perpendicular from the point *b* on the plane *pqr* (Prob. XV.). This perpendicular is therefore the intersection of these planes, and is consequently perpendicular to the line joining *e* and *d* in space, and likewise to that joining *a* and *c*. Now *b*F is the perpendicular from *b* on the line joining *e* and *d* in space, turned down upon the horizontal plane; and *b*G is the perpendicular from *b* on the line joining *a* and *c* in space, similarly turned down. Hence *b*F and *b*G being two measures of the same line, are equal; and *b*H is equal to *b*C, and the angles at F and G are right angles; consequently the angle F*b*H is equal to G*b*C, and therefore to *ba*C; that is, the angle F*b*H is equal to the angle of inclination of the plane *pqr* to the horizontal plane.

Problem XXII.

Through a given point, to draw a plane having given angles of inclination to the planes of projection.

The corollary to the last problem affords a ready solution to this, by enabling us, in the first place, to construct a plane having the given angles of inclination to the planes of projection: a plane parallel to this, and passing through the given point, will be the plane required.

Let [*m*, *m'*] (fig. XXII.) be the given point, and α, β the given angles of inclination of the plane to the horizontal and vertical planes of projection. In *xy* take any point D, and make the angle *y*DH equal to the angle β which the plane is to make with the vertical plane; and through any other point *b* in *xy* draw *cbe* perpendicular to *xy*, and from *b* draw *b*F perpendicular to D*e*. At *b* draw *b*H, making the angle F*b*H equal to the angle α which the plane is to make with the horizontal plane, and meeting D*e* in H. From the centre *b* at the distance *b*D describe a circle; from the same centre at the distance *b*H describe a circle cutting *bc* in *c*; and from *c* draw *cdq* touching the former circle, and meeting *xy* in *q*: join *qe*. The plane *cqe* makes the angles α, β with the planes of projection. For the angle *b*D*e*, which is equal to β, is the angle that the line joining *d* and *e* in space makes with *bd*, or the angle of inclination of the plane *cqe* to the vertical plane, turned down upon the horizontal plane. Also drawing *b*C perpendicular to *ab*, and joining *a*C, the angle F*b*H, which is equal to α, is equal to the angle *ba*C (Prob. XXI. Cor.), which is the angle that the line joining *a* and *c* in space makes with *ba*, or the angle of inclination of the plane *cqe* to the horizontal plane, turned down upon the horizontal plane.

Through the point [m, m'] draw the plane stv parallel to the plane pqr (Prob. IX.): stv is the plane required; for parallel planes have the same inclination to a plane which cuts them (Prop. 40).

PROBLEM XXIII.

Through a given straight line, to draw a plane having a given angle of inclination to the horizontal plane.

When the given line cuts both planes of projection, its traces on these planes will be points in the traces of the plane required; and if another point in either trace can be found, both traces will be determined.

Let [ab, $a'b'$] (fig. XXIII.) be the given straight line, and LMN the given angle. From a and b' draw aa' and $b'b$ at right angles to xy, so that b and a' are the traces of the given line (Prob. I.). Draw LN at right angles to MN; and at the point a' make the angle $aa'm'$ equal to the angle NLM, so that the angle $a'm'a$ is equal to the given angle LMN. From the centre a, at the distance am', describe the circle $m'pq$; and from b draw bpt touching the circle $m'pq$, and meeting xy in t. Join $a't$; $a't$, tb are the traces of the plane required.

Since aa' is perpendicular to the plane xby, and ap is at right angles to bt (III. 16), the line joining a' and p, in space, is perpendicular to the trace tb (Prop. 16). Consequently the angle formed by this line with pa is the inclination of the plane $a'tb$ to the horizontal plane. The triangle formed by this line with pa and aa' is right-angled at a, as is the triangle $a'am'$; and pa is equal to $m'a$, and aa' common to this triangle and the triangle $m'aa'$: therefore the angle formed at p with pa, by the line joining a' and p in space, that is, the inclination of the plane $a'tb$, is equal to the angle $a'm'a$, that is, to the given angle LMN.

The angle formed by the line joining a' and p in space, with pa, may also be shown to be equal to the angle $am'a'$, by conceiving the plane paa' to turn about aa' until it coincides with the vertical plane, when this angle will coincide with $a'm'a$.

The given angle LMN may be such that am' becomes equal to bb'. In this case, the horizontal trace of the plane required is parallel to the line of level xy, and therefore the vertical trace is also parallel to xy (Prop. 23).

Other particular cases are:

1. When the given line is parallel to the horizontal plane. In this case, its vertical projection $a'b'$ (fig. XXIII. 1) is parallel to xy (35. 2), and the horizontal trace of the plane required will be parallel to the horizontal projection ab (Props. 23, 19) of the line. The only difference in this case is, that the horizontal trace of the plane is found by drawing ut parallel to ab, to touch the circle $m'pq$. The vertical trace is found as before by joining t and the vertical trace a' of the given line.

2. When the given line is parallel to the vertical plane. In this case, its horizontal projection ab (fig. XXIII. 2) is parallel to xy, and the vertical trace of the plane required will be parallel to the vertical projection $a'b'$ of the given line (Props. 23, 19).

At any point n', in $b'y$, draw $n'l$, making the angle $b'n'l$ equal to the given angle NML, and meeting $b'l$, perpendicular to xy, in l; and through l draw ls parallel to $a'b'$, meeting xy in s. From b' as a centre, at the distance $b'n'$, describe the circle $n'rq$: and from s draw sru touching the circle, and meeting $b'b$ in u. Drawing uw parallel to xy, uw and $b'a'$ will be the horizontal and vertical projections of a line parallel to the given line, and whose horizontal trace is u; and from what has been stated, ls and su will be the traces of a plane passing through that line and making an angle with the horizontal plane equal to $ln'b'$, that is, to the given angle LMN. Therefore drawing bt parallel to us, and through t, tv' parallel to ls or $a'b'$; bt and tv' are the traces of a plane parallel to the former plane (Prob. IX.), and therefore making an angle with the horizontal plane equal to the given angle, and also passing through the given line whose horizontal trace is b.

3. When the given line is parallel to both planes of projection. In this case, both projections of the given line are parallel to the line of level (35. 2), and both traces of the plane required will also be parallel to this line (Props. 23, 19).

Draw any line pr' (fig. XXIII. 3), perpendicular to xy, and therefore to the projections $a'b'$, ab of the given line, cutting them in a', a and xy in m. In ab take an equal to ma', and in $a'b'$, $a'n'$ equal to ma. At n, draw np, meeting $r'p$ in p, and making the angle anp equal to the angle MLN, and therefore the angle apn equal to the angle LMN; and at n' draw $n'r'$ meeting pr' in r', and making the angle $a'n'r'$ equal to the angle LMN. Through p and r' draw pq, $r's'$ parallel to xy: these are the traces of the plane required.

For the plane $xa'y$ being in its vertical position, conceive the triangle $a'n'r'$ to turn about $a'r'$, and at the same time the triangle apn to turn about ap until both these triangles are in the vertical plane passing through ama'. Then an being equal to ma', and $a'n'$ equal to ma, the points n and n' will coincide in the point A of the given line, of which a and a' are the projections; the lines pn, $n'r'$ will form one straight line in the plane whose traces are pq and $r's'$, because the angles on one side of pn and $n'r'$ at the point of meeting are together equal to two right angles; and this plane will pass through the given straight line. And since mp, and the continued line formed by pn and $n'r'$ are both at right angles to the horizontal trace pq, the angle which the plane whose traces are pq, $r's'$ makes with the horizontal plane is the angle apn, that is, is equal to the given angle LMN.

Problem XXIV.

To find the angle made by two given straight lines; that is, the projections of two straight lines being given, to construct the angle formed by these lines in space.

If the horizontal and vertical projections of the two given straight lines do not meet in the same perpendicular to xy, the lines in space do not meet, and the angle which they make with each other is then considered to be the angle formed by one of them with a parallel to the other, meeting the first. It is, therefore, the angle formed by the two given straight lines, when they meet, or the angle formed by one of them and a parallel to the other meeting the first, that we have to construct.

When both the intersecting straight lines meet the horizontal plane, the angle formed by them is the vertical angle of a triangle of which the base is the distance between the horizontal traces of the lines, and the sides are the distances of these traces from the point where the lines intersect in space. It is therefore this angle which we have to construct.

Let ab and $a'b'$ (fig. XXIV.), ac and $a'c'$ be the projections of the two given lines, or of one of them and the parallel to the other, which meet in the point of which a and a' are the projections. Drawing $b'b$ and $c'c$ perpendicular to xy, b and c are the horizontal traces of the lines; and joining bc, this line with the two given lines in space will form a triangle of which the angle opposite to bc is the required angle, formed in space by the two given straight lines at their point of intersection A, of which a and a' are the projections; that is, the required angle is the angle at the point $[a, a']$ in the triangle b $[a, a']$ c. Drawing ad perpendicular to bc, the line joining d and the point $[aa']$ or A, in space, will be perpendicular to bc (Prop. 16), and will be the hypothenuse of a right-angled triangle of which the base is ad, and perpendicular ma'. Taking, therefore, md'' equal to ad, $a'd''$ is the distance from d to the point $[a, a']$, or the length of the line d $[a, a']$. Since the plane da $[a, a']$ is perpendicular to bc (Prop. 4), if the triangle b $[a, a']$ c turn about bc until it coincide with the horizontal plane, the line d $[a, a']$ will come down upon da, the point $[aa']$ coinciding with a point $A_{\prime}$ in da produced, so that $dA_{\prime}$ is equal to $d''a'$; and the angle $bA_{\prime}c$ will be the required angle b $[a, a']c$ thus brought down upon the horizontal plane.

In the construction of the triangle of which the vertical angle is required, we have made use of the base and the perpendicular: we might have employed the sides. Since ba, ca are the horizontal projections of the sides of the triangle, and ma' is the height of their point of intersection above the horizontal plane, the sides of the triangle are the hypothenuses of right-angled triangles of which ba, ca are the bases, and ma' is the perpendicular. Taking, therefore,

mb'' equal to ab, and mc'' equal to ac, $a'b''$ and $a'c''$ are the required sides; and if from b and c, as centres, and at distances equal $a'b''$ and $a'c''$ respectively, arcs are described, their intersection $A_{,}$ will be the vertex of the triangle, and the angle $bA_{,}c$ the angle required.

Particular cases.—1. The general construction assumes that the two lines intersect the horizontal plane. If one of the lines [ac, $a'c'$] (fig. XXIV. 1) be parallel to this plane, its vertical projection $a'c'$ will be parallel to xy (35. 2), and the horizontal trace bf of the plane containing the angle will be parallel to the horizontal projection ac of the line (Prop. 23, 19). The angle may still be brought down upon the horizontal plane by means of the perpendicular ad, the only difference from the general construction being that the horizontal side will remain parallel to bf or ac, having the position $A_{,}C_{,}$.

If both the given lines be parallel to the horizontal plane, it is evident (Props. 23, 20) that the angle formed by their horizontal projections will be equal to that formed by the lines themselves.

2. The general construction also assumes that the angular point is out of the horizontal plane. When that point is in the horizontal plane, the horizontal traces of the two lines [ab, $a'b'$], [ac, $a'c'$] are in the same point a. In this case, a straight line [$b\gamma$, $b'\gamma'$] (fig. XXIV. 2) being drawn through any point [b, b'] in one of the lines [ba, $b'a'$] parallel to the other [ca, $c'a'$], the angle made by the line [ba, $b'a'$] with the line [$b\gamma$, $b'\gamma'$], at the point [b, b'], will be equal to the angle made by it at the point [a, a'], with the line [ca, $c'a'$] (I. 29). Constructing as in the figure, according to the general case, the angle $\gamma A_{,}a$ is equal to the former of these angles, and consequently to the latter, which is required.

Problem XXV.

To construct the angle made by a given straight line with a given plane.

If from any point in the given straight line a perpendicular be let fall upon the given plane, the angle contained by these lines will be the complement of the angle of inclination of the given line to the plane. We have therefore only to construct the angle contained by these two lines.

Let ab, $a'b'$ (fig. XXV.) be the projections of the given straight line, and $m\alpha$, $m'\alpha$ the traces of the given plane. Take a and a' in the projections of the given line, and in the same perpendicular to xy, so that they are the projections of the same point; and from them draw ac and $a'c'$ perpendicular to $m\alpha$ and $m'\alpha$ respectively: these are the projections of the perpendicular from the point [a, a'] on the plane (Prob. XV.). Drawing $b'b$ and $c'c$ perpendicular to xy, b and c are the horizontal traces of the given straight line and the perpendicular from a point in it on the given plane. Constructing as in the figure, according to Problem XXIV., $bA_{,}c$ is equal to the

angle contained by the given straight line and the perpendicular to the given plane; and drawing $A_{\prime}h$ at right angles to $cA_{\prime}$, $bA_{\prime}h$ is equal to the angle of inclination of that straight line to this plane.

Problem XXVI.

To construct the angle of inclination of two given planes.

If a plane be drawn perpendicular to the intersection of the two planes, its traces on these planes will be perpendicular to the intersection (Def. 3), and therefore the angle contained by these traces will be the inclination of the planes (Def. 7). This angle being brought down upon one of the planes of projection will be the angle required.

Let $a\alpha a'$, $a\beta a'$ (fig. XXVI.) be the two given planes; then a and a' are the traces of their intersection; and $a'b$ being perpendicular to xy, ab is the horizontal projection of this intersection. Draw any line gch perpendicular to ab, as the horizontal trace of a plane perpendicular to the intersection of the two planes, that is, to the line joining aa' in space. The traces of this plane on the given planes are perpendicular to that line, and pass, the one through g, the other through h, forming a triangle of which gh is the base; and the vertical angle of this triangle, being contained by these traces, is the angle which it is required to construct. The vertex of this triangle is in the vertical plane aba'; but the line gh is perpendicular to this plane (Def. 6); therefore the line joining c and the vertex of the triangle is perpendicular to gh (Def. 3); and consequently the triangle being brought down upon the horizontal plane, by turning upon gh as an axis, this line will be in the direction ca. We have therefore only to find the true length of this line in order to construct the angle.

The plane of the triangle being perpendicular to the line joining a and a' in space, the line joining c and the vertex is perpendicular to this line. Making then the vertical plane aba' turn about its vertical trace $a'b$, until ba coincides with by, the points a and c will describe arcs of circles ap, cq, about the centre b, and have the positions p and q; and the intersection of the two given planes will be upon $a'p$. Drawing therefore qr perpendicular to $a'p$, this will be the true length of the perpendicular from the vertex to the base gh of the triangle. Consequently taking cN equal to qr, the angle gNh is the angle of inclination of the two planes.

PROBLEM XXVII.

Through a given straight line, to draw a plane that shall have a given angle of inclination to a given plane.

The several cases of this problem are:

1. *When the given straight line is in the given plane.*

The construction in the last problem immediately furnishes the solution of this, which is its converse.

Let $a\alpha a'$ (fig. XXVII. 1) be the given plane, and a, a' the traces of the given line in it. Draw $a'b$ perpendicular to xy, and join ab; then ab is the horizontal projection of the given line joining a and a' in space. From any point h in $a\alpha$ draw hc perpendicular to ab, and produce it if necessary. From the centre b, at the distances ba, bc describe circles cutting xy in p, q; join $a'p$, and from q draw qr perpendicular to it: then from what has been stated in the last problem, qr is the length of the line drawn from c perpendicular to the line joining a, a' in space. Take cN equal to qr; join hN; and at the point N draw Ng to meet hc, produced if necessary in g, making the angle hNg equal to the given angle of inclination of the planes. Join ag, and produce it to meet xy in β; and join $a'\beta$: the plane $a\beta a'$ is the plane required. For by the last problem the angle hNg is the angle of inclination of the plane $a\beta a'$ to the plane $a\alpha a'$; and the angle hNg being made equal to the given angle of inclination, the plane $a\beta a'$, passing through the given line whose traces are a, a', has the required inclination to the given plane $a\alpha a'$.

2. *When the given straight line is parallel to the given plane.*

In this case, straight lines drawn through the traces of the given straight line, parallel to the traces of the given plane, are the traces of a plane passing through the given line, and parallel to the given plane; and a plane drawn, by the first case, through the given line to have the given angle of inclination to this parallel plane, will also have the given angle of inclination to the given plane (Prop. 40).

It is unnecessary here to go through the construction of this case with a figure, but the student is recommended to do so, as an exercise.

3. *When one of the traces of the given straight line is in one of the traces of the given plane.*

Let the vertical trace of the given straight line be in the vertical trace of the given plane. If from the horizontal trace of the given line, a perpendicular be drawn to the given plane, and also a straight line making the given angle of inclination with that plane, and a circle be described on that plane from the foot of the perpendicular as a centre, at the distance of the intersection of the oblique line; then a plane passing through any tangent to the circle, and the horizontal trace of the given line, will make the given angle of inclination with the given plane (Prop. 16 and Def. 7). If, therefore,

from the vertical trace of the given line a tangent be drawn to the circle, the plane passing through this tangent and the horizontal trace of the given line will pass through this line, and make the given angle with the given plane. This tangent to the circle thus described on the given plane will be the trace on that plane of the plane required, and therefore to determine this trace we have only to effect the above construction on one of the planes of projection.

Let $\alpha a a'$ (fig. XXVII. 3) be the given plane, [bc, $b'c'$] the given straight line, and LMN the given angle of inclination. Draw cm, $c'n'$ perpendiculars to αa, $\alpha a'$, and determine by Problem XV., p and p' the projections of the foot of the perpendicular from the point [c, c'] upon the plane $a\alpha a'$. The length of the perpendicular is the hypothenuse of a right-angled triangle of which the sides are its projections cp and qp'. Drawing therefore pP perpendicular to cm, and equal to qp', joining cP, cP is the length of the perpendicular from c upon the plane $a\alpha a'$. Join mP. If now the triangle cPm be conceived to turn about the side cm until its plane is perpendicular to the horizontal plane, Pp, being perpendicular to cm, will, in this position, be perpendicular to the horizontal plane; and since pP is equal to qp', P will be the point of which p and p' are the projections, that is, the foot of the perpendicular from the horizontal trace c of the given line, on the plane $a\alpha a'$, in its true position: cP will be the perpendicular in its true position; and mP, being a line meeting cP in the plane, will be perpendicular to it, or make the angle mPc a right angle. Draw cR making the angle PcR equal to MLN, and therefore the angle cRP equal to the given angle of inclination LMN. If therefore in this position of the triangle cPm, a line be drawn in the plane $a\alpha a'$, through R perpendicular to Pm, it will also be perpendicular to cR (Prop. 16); consequently a plane passing through this perpendicular and Rc will have the required angle of inclination LMN to the plane $a\alpha a'$ (Def. 7).

Produce mc to meet xy in e, and draw ee' perpendicular to xy, meeting $\alpha a'$ in e'. Restoring now the triangle mPc to its original position in the horizontal plane, conceive the plane $a\alpha a'$ also to be brought down upon that plane by turning about the line αa. The point e' will describe a circle about α as a centre; the line of which me is the horizontal projection, being at right angles to αa, will be brought down upon mE, perpendicular to αa, or in em produced, and its intersection E with the circle described by e' will be the position of this point on the horizontal plane; and αE will be the vertical trace of the given plane brought down upon the horizontal plane, by the former plane turning about αa. Also the point b', which is the vertical trace of the given line, will describe a circle and be in the point B, making αB equal to $\alpha b'$.

From the centre m at the distances mP, mR describe circles intersecting mE in Q, V; then, from the centre Q, at the distance QV,

describe the semicircle VDFS; and from B draw BDT, BFW touching this circle in D, F, and meeting mE in T, W.

Conceive the plane $m\alpha$E to be restored to its true position by revolving about αm, so that αE coincides with $\alpha e'$, when the plane $e\alpha e'$ is likewise in its true position, that is, vertical. Also conceive the plane cmG again to become vertical, by revolving about cm. The points Q, T, W will describe circles about the centre m, and be found at the points P, H, G in the plane cmG, mP being equal to mQ, mH equal to mT, and mG equal to mW. In this position cP, that is cQ, being perpendicular to the plane $a\alpha a'$, that is, $m\alpha$E, QD perpendicular to BT, and QF to BW, the line joining c and D in space will also be perpendicular to BT (Prop. 16), and the line joining c and F in space will be perpendicular to BW. Since the lines joining c and D, c and F make the same angles with the plane $a\alpha a'$ which the line joining c and V, or cR does, it follows from what has already been stated that the plane passing through BT and c, and also that through BW and c will have the angle of inclination to the plane $a\alpha a'$ equal to the angle cRP, that is, to the given angle LMN. BT and BW are therefore the traces on the given plane, of planes having the required angle of inclination to it; and it remains only to determine the horizontal traces of these lines of which the vertical trace is b, in order to obtain the traces of the latter planes.

Since the lines BT, BW are in the plane $a\alpha a'$, their horizontal traces will be in $a\alpha$, the horizontal trace of that plane; consequently, the plane $a\alpha a'$ being again turned down upon the horizontal plane, as $a\alpha$E, producing BT and WB to meet $a\alpha$ in l and o, these are the horizontal traces of the lines BT and BW in space, and are therefore points in the horizontal traces of the planes passing through these lines. Drawing therefore $lc\beta$, $o\gamma c$ through c, the horizontal trace of the given line, these are the horizontal traces of the planes having the given angle of inclination LMN to the given plane $a\alpha a'$, the vertical traces of these planes being $b'\beta$, $b'\gamma$.

4. *When the given straight line is not parallel to the given plane, and neither of its traces is in a trace of that plane.*

In this case, if a plane be drawn parallel to the given plane, and having its vertical trace passing through that of the given line, then a plane drawn, by the last case, passing through the given line and having the given angle of inclination to this parallel plane, will also have the given inclination to the given plane (Prop. 40).

The construction of this case is left as an exercise for the student.

Remarks.—1. The traces of the planes in the third case may be determined by first determining the projections of the lines BT, BW, and thus obtaining their horizontal traces, but the method which has been given is much more simple. The student may, however, adopt that method as an exercise, and also as a verification of the other.

2. The distinction between the two planes which thus make a given dihedral angle with the given plane is this, that the lower dihedral angle which is turned towards the horizontal trace of the given plane is in the one case obtuse, and in the other acute.

3. When PR is equal to QB, the circle described upon the plane $\alpha\alpha a'$, from the centre Q or P, at the distance PR, passes through the point B or b'; the tangents to it drawn from that point are in one straight line, to which QB or Pb' is at right angles; the given line, joining c and b' in space, is itself the perpendicular from c on this tangent to the circle; and therefore the two planes having the required angle of inclination to the plane $\alpha\alpha a'$ coincide. Also, since the given line makes a less angle with the plane $\alpha\alpha a'$, than any straight line drawn from c to a point in that plane nearer to the perpendicular on it than B or b', the angle formed with the plane $\alpha\alpha a'$ by the plane passing through the given line and the line which is at right-angles to QB or Pb', in the plane $\alpha\alpha a'$, is less than the angle formed with that plane by any other plane passing through the given line. The limit of the problem therefore is, that the given angle of inclination shall not be less than that of the given straight line to the given plane.

Problem XXVIII.

To reduce to the horizontal plane, the angle contained by two straight lines.

When, in the survey of a country, the angles which are observed between objects whose positions are to be determined, are not horizontal angles, in order to lay down the positions of those objects in a plan, it is necessary that the observed angles should be reduced to the horizontal plane; that is, that from the observed angle contained by two straight lines drawn from any station to two objects, the angle contained by the intersections of the horizontal plane with two vertical planes passing through the two straight lines, or the angle formed by their horizontal projections, should be determined. In order to do this, it is necessary that, besides the angle contained by the two lines, the angles which these lines make with the vertical should be observed.

If we conceive vertical planes to pass through the lines containing the given angle, their intersection will be perpendicular to a horizontal plane (Prop. 32). The height of the angular point of the given angle above this horizontal plane being assumed, and the angles which the lines containing that angle make with the vertical line passing through the angular point being known, the horizontal distances from this vertical at which these lines meet the horizontal plane will be determined, since they are the bases of right-angled triangles of which the perpendicular and vertical angles are given;

and the lengths of the lines containing the given angle, from the angular point to the horizontal plane, will also be determined, since they are the hypothenuses of the same right-angled triangles. The latter two lines and the given angle contained by them will determine a triangle of which the base is horizontal; and this base with the former two horizontal lines will determine a horizontal triangle, of which the angle opposite to this base is the given angle reduced to the horizontal plane, for it is the angle contained by the intersections of that plane with the vertical planes passing through the lines containing the given angle. On these principles we have the following construction.

To render the references in this construction clearer, we will call the given angle A; the two lines containing that angle, from the angular point to their intersection with the horizontal plane, L and L′; the angles which these lines make with the vertical, V and V′, *a* the angle A reduced to the horizon; and *l*, *l′* the horizontal lines containing that angle, from the intersection of the vertical through the angular point with the horizon, to the intersection of L and L′ with the same plane. Let *nm* (fig. XXVIII.) be perpendicular to *xy*, and *n* be the angular point of the given angle. Make the angle *mnp* equal to V, and *mnq*=V′; then *np* and *nq*, being the hypothenuses of right-angled triangles of which the perpendicular is the height of *n* above the horizontal plane, and the vertical angles are V, V′, are equal to L and L′ respectively; and *mp* and *mq*, being the bases of the same right-angled triangles, are equal to *l* and *l′*, the sides which contain the horizontal angle *a*. At *n* make the angle *pnr* equal to A; and from *n* as a centre, at the distance *nq* describe a circle cutting *nr* in *r*; and join *pr*: *pr* is equal to the base of the triangle whose sides are *l* and *l′*, and their included angle is *a*. To construct this triangle, we have only to describe a circle from *m* as a centre, at the distance *mq*, and another from *p* as a centre, at the distance *pr*, intersecting the former in *s*; joining *ms*, *ps*; *mps* is evidently the triangle of which the sides are *l*, *l′*, and *pr*; and *pms* is the required horizontal angle *a* included by *l* and *l′*.

MISCELLANEOUS PROBLEMS.

Problem XXIX.

The position of a point on a given plane being given, to determine its horizontal and vertical projections.

In order that the point may be represented in its true position with reference to the traces of the given plane, it is necessary that this plane should be turned down upon one of the planes of projection, the horizontal for example. Its position on the given plane being then determined by a parallel to the vertical trace thus

brought down, and another parallel to the horizontal trace, if the plane be replaced in its original position, the projections of these parallels will determine the projections of the given point.

Let *rst* (fig. XXIX.) be the given plane. Take any point n' in *st*; draw $n'n$ perpendicular to *xy*; and *n*N perpendicular to *sr*. From *s* as a centre, at the distance sn' describe a circle cutting *n*N in N; draw *s*N : *rs*N is the given plane turned down upon the horizontal plane (Prob. XXI.). Let A be the given point in this plane; draw A*p* parallel to SN, and AQ to *sr*. The plane *rs*N being replaced in its original position, *p*A being parallel to the vertical plane of projection, the vertical plane passing through it will also be parallel to that plane, and therefore its horizontal projection *pa* will be parallel to *xy* (Prop. 27); and AQ being parallel to the horizontal plane of projection, its vertical projection will, for the same reason, also be parallel to *xy*. Also *p*A being parallel to *st*, its vertical projection will likewise be parallel to *st*; and AQ being parallel to *sr*, its horizontal projection will be so likewise. In restoring the plane *rs*N to its original position, the point Q will describe a circle about *s* as a centre, and, making sq' equal to sQ$'$, will be found at q'. Drawing, therefore, *pa* and $q'a'$ parallel to *xy*; pp' and $q'q$ perpendicular to *xy*; $p'a'$ parallel to *st*, and *qa* parallel to *sr*; *pa* and $p'a'$ are the horizontal and vertical projections of *p*A; *qa* and $q'a'$ are those of QA; and *a* and a' are those of the point A, the lines *p*A, QA, and the point A being on the plane *rst* in its true position.

If the construction be correct, the straight line $a'a$ will be perpendicular to *xy*; also the straight line joining A*a* will be perpendicular to *sr*. These verifications afford the means of conveniently varying the construction according to circumstances.

Problem XXX.

From the true representation of any plane rectilineal figure upon a given plane, to determine the horizontal and vertical projections of that figure.

The given plane being turned down upon the horizontal plane, and the figure drawn on it, in its true dimensions and position relatively to the traces of the given plane, the projections of the figure will be determined by obtaining the projections of its angular points, by a repetition of the processes described in the last problem, or those referred to in the remark.

ABCDE (fig. XXX.), a symmetrical but not a regular pentagon, is the true representation, on the plane *rst* turned down upon the horizontal plane of projection, of the rectilineal figure whose horizontal and vertical projections are required. The horizontal projection of the point A, *a*, is determined by drawing A*a* perpendicular to *sr*, A*p* parallel to *s*N, and *pa* parallel to *xy*; and its vertical

projection a', by drawing aq parallel to rs, aa' and qq' perpendicular, and $q'a'$ parallel to xy, in conformity to the remark in the last problem. The projections of the other points in the figure are determined by a repetition of this process.

When rs is perpendicular to xy, or the plane rsn perpendicular to the vertical plane, the above method of determining the horizontal projection fails, Ap and pa being in that case perpendicular to sr. In that case the point q' being determined as in Problem XXIX., by taking sq' equal to sQ, a' the vertical projection of A will coincide with q', and the horizontal projection a will be determined by the intersection of the perpendicular on xy, from a' or q' with pa parallel to xy. The projection of the other points B, C, D, E are similarly determined, their vertical projections being all in the line sr.

Problem XXXI.

From the horizontal projection of a plane rectilineal figure, wholly in a given plane, to construct the figure on that plane; and to determine its vertical projection.

This problem being the converse to the last, we obtain the required construction by reversing the processes in that problem.

Let $abcde$ (fig. XXXI.) be the horizontal projection of the rectilineal figure on the given plane rst. This plane being, as before, turned down upon the horizontal plane, in rsN, from a draw aA perpendicular to rs, and ap parallel to xy; and from p draw pA parallel to sN: A is the true position on the plane rsN of the point in rst, of which a is the horizontal projection. The points B, C, D, E being similarly determined, ABCDE is the true representation of the figure on the plane rsn', of which $abcde$ is the horizontal projection.

The vertical projection of ABCDE is determined from its horizontal projection as in the last problem.

Remarks.—1. When sr is perpendicular to xy this construction fails. In this case the vertical projection corresponding to a will be the intersection with st of the perpendicular to xy drawn from a; and the point A will be the intersection of the parallel to xy, drawn from a, with a parallel to sr, or a perpendicular to xy, drawn at a distance from s equal to sq' or sa'.

2. By the former part of this construction we obtain the section of any right prism, made by a plane inclined at a given angle to the base, and passing through a straight line given in position relatively to the base. For the vertical lines drawn from a, b, c, d, e, and passing through A, B, C, D, E, when the plane rsn is turned back into its true position, are the edges of a right prism of which the base is $abcde$, and the opposite end a figure similar and equal to

abcde; and ABCDE is the section of this prism by a plane passing through *sr*, and making a given angle with the horizontal plane or that of the base *abcde*. Whatever therefore may be the position of the base of the prism, we have only to consider that plane as representing the horizontal plane in this problem. Thus in fortification, from the section of a work made by a vertical plane perpendicular to its face, we may immediately determine its section by a vertical plane making a given angle with the face, by considering the former section as the base of a right prism, the second section being made by a plane inclined to this base at a given angle.

Problem XXXII.

From the horizontal projection of the diagonal of a square which is on a given plane, to determine the horizontal and vertical projections of the square.

From the given horizontal projection of the diagonal of the square, we obtain by the last problem the construction of this diagonal on the given plane turned down upon the horizontal plane. Constructing with this line the square of which it is the diagonal, we immediately obtain the horizontal and vertical projections of this square by Problem XXX.

Thus *ab* (fig. XXXII.) being the given horizontal projection of the diagonal of a square on the plane *rst*, constructing as in the last problem, AB is this diagonal on the plane *rst* turned down upon the horizontal plane. Constructing the square ABCD on AB as a diagonal, then by Problem XXX., we have *acbd* the horizontal projection of the square ACBD, and $a'c'b'd'$ its vertical projection.

Problem XXXIII.

From a given point in a given plane, to draw a perpendicular to the plane, which shall be of a given length; that is, to find the projections of the point in the perpendicular to the plane, which is at a given distance from the plane.

The projections of the perpendicular to the plane, from the given point in it, are perpendiculars to the traces of the plane, drawn from the projections of that point (Prob. XV.). Assuming any point in the perpendicular as its further extremity, the true length of this perpendicular may be determined by Problem XV. We have, therefore, to find this point such that the length thus determined shall be the given length.

Let *f* (fig. XXXIII.) be the horizontal projection of the given point in the given plane *rst*; and (drawing *fp* parallel to *rs*, pp' and ff' perpendicular, and $p'f'$ parallel to *xy* (Prob. XXIX.)), f' its vertical projection. Then *fg*, $f'g'$, perpendiculars to *rs*, *st*, are the pro-

jections of the indefinite perpendicular to the plane, drawn from the point [f, f']. Taking any point [g, g'] in this perpendicular; drawing, through f', jk parallel to xy, cutting $g'g$ in h; and taking hi equal to fg; $g'i$ is the true length of the perpendicular to the plane drawn from the point [g, g'] to the point [f, f'] in the plane. Take $f'l'$ a fourth proportional to ig', the given length of the perpendicular, and $f'g'$; and draw $l'j$ parallel to ig', cutting $f'i$ produced in j; jl' is the given length of the perpendicular; for $f'g' : f'l' :: ig' : jl'$. Draw $l'l$ perpendicular to xy or parallel to $g'g$: l and l' are the projections of the point in the perpendicular, which is at the given distance from the point [f, f']. For

$$
\begin{array}{ll}
 & f'i : f'j :: f'g : f'l' \\
 & \qquad\quad :: f'h : f'k \\
\text{alternando} & f'i : f'h :: f'j : f'k \\
\text{componendo} & ih : f'h :: jk : f'k \\
\text{alternando} & ih : jk :: f'h : f'k \\
 & \qquad\quad :: fg : fl;
\end{array}
$$

but ih is equal fg, therefore jk is equal to fl: consequently jl' is the true length of the perpendicular from the point [l, l'] to the plane rst (Prob. XV.): that is, the point [l, l'], in the perpendicular [lf, $l'f'$], is at the given distance from the point [f, f'], in the plane rst.

Problem XXXIV.

From the altitude and given position of the base of any right prism standing upon a given plane, to determine the horizontal and vertical projections of the prism.

From the position of the base of the prism on the given plane, its horizontal and vertical projections are determined by Problem XXX. If, therefore, from the projections of the angular points of the base, the projections of perpendiculars to the plane, equal to the given altitude of the prism, be drawn by the last problem, these will be the projections of the perpendicular edges of the prism; and their extremities being joined, will give the projections of the other end of the prism.

As an example, the projections of a cube standing upon a given plane inclined to the co-ordinate planes are given in figure XXXIV. From the position of the base ACBD on the given plane rst turned down upon the horizontal plane, its horizontal and vertical projections $acbd$, $a'c'b'd'$ are determined by Problem XXX. The projections of a perpendicular to the plane rst drawn from the point s and equal to AC, one edge of the cube, are determined by the last problem, sl being its horizontal, and sl' its vertical projection. Drawing, therefore, au, cw, bv, dz perpendicular to rs, and equal to sl; and joining uw, wv, vz, zu; $abwz$ is the horizontal projection of the cube standing

on the plane *rst*: and again, drawing $a'u'$, $c'w'$, $b'v'$, $d'z'$ perpendicular to *st*, and equal to sl'; and joining $u'w'$, $w'v'$, $v'z'$, $z'u'$; $a'b'w'z'$ is its vertical projection.

Problem XXXV.

From the altitude and given position of the base of any pyramid standing upon a given plane, to determine the horizontal and vertical projections of the pyramid.

The construction here is almost the same as in the last problem, the only difference being that the projections of the vertex of the pyramid are to be determined; these projections and those of the angular points of the base being respectively joined, will give the projection of the pyramid. The projections of the vertex will be determined from the projections of the perpendicular let fall from that point upon the base.

As an example, the projections of a square pyramid having equilateral faces, standing upon a given plane inclined to the co-ordinate planes, are given in figure XXXV.; and in the same figure are given the projections of an equal pyramid having its base applied under the given plane, to that of the former pyramid, so that the whole figure represents the projections of a regular octahedron having its axis bisected perpendicularly by the given plane.

ABCD is the base of the pyramid, or the section of the octahedron at right angles to the middle of its axis, on the given plane *rst* turned down upon the horizontal plane, and its horizontal and vertical projections *acbd*, $a'c'b'd'$ are determined as before, by Problem XXX. The perpendicular from the vertex to the base of the pyramid being the side of an isosceles right-angled triangle of which the hypothenuse is equal to a side of the base of the pyramid, the projections of a perpendicular to the plane *rst*, drawn from the point *s* and equal to this perpendicular, are *sl*, sl', determined by Problem XXXIII. From *e*, the middle point of *acbd*, *ev* is drawn perpendicular to *rs*, and equal to *sl*; and from e', the middle point of $a'c'b'd'$, $e'v'$ is drawn perpendicular to *st*, and equal to sl'; joining *av*, *cv*, *bv*, *dv*, *acvbd* is the horizontal projection of the pyramid whose base is ACBD standing upon the plane *rst*; and joining $a'v'$, $c'v'$, $b'v'$, $d'v'$, $a'c'v'd'b'$ is its vertical projection.

Producing *ve*, making *ez* equal to *ev* or *sl*, and joining *az*, *cz*, *bz*, *dz*, *vaczbd* is the horizontal projection of the octahedron whose axis is perpendicular to the plane *rst*; and producing $v'e'$, making $e'z'$ equal to $e'v'$ or sl', and joining $a'z'$, $c'z'$, $b'z'$, $d'z$, $v'a'c'z'b'd'$ is its vertical projection.

Problem XXXVI.

To construct the shortest distance between two given straight lines not in the same plane.

The principles on which the construction of this problem depends have been already given in Proposition XXXVI. of the Geometry of Planes.

Let AB, CD (fig. XXXVI. *a*) represent the given straight lines. Through AB draw the plane MN parallel to CD; from the point P, in CD, let fall the perpendicular PR on the plane MN; from R draw RS parallel to CD meeting AB in S; and from S draw ST parallel to PR, and therefore perpendicular to MN (Prop. 18), meeting CD in T: ST, which is perpendicular both to CD and AB, is the shortest distance between these lines. We have, therefore, only to effect these constructions on the planes of projection.

Let ab, $a'b'$ (fig. XXXVI.) and cd, $c'd'$ be the projections of the given straight lines. Find the traces a, b' of $[ab, a'b']$ (Prob. I.). Through o, o' the projections of a point in the line $[ab, a'b']$ draw mn, $m'n'$ parallel to cd, $c'd'$; from m' and n draw $m'm$ and nn' perpendicular to xy; and draw the lines $am\alpha$, $n'b'\alpha$: these are the traces of the plane passing through $[ab, a'b']$ parallel to $[cd, c'd']$ (Prob. XII.).

From p, p', the projections of a point in the line $[cd, c'd']$, draw pq, $p'q'$ perpendicular to $a\alpha$, $b'\alpha$: these are the projections of a straight line drawn from the point $[p, p']$, perpendicular to the plane $a\alpha b'$ (Prob. XV.). Determine, by Problem XV., the projections r, r' of the intersection of this perpendicular with the plane $a\alpha b$; and draw rs, $r's'$ parallel to cd, $c'd'$, meeting ab, $a'b'$ in s, s': these points are the projections of the intersection with the line $[ab, a'b']$ of the straight line drawn in the plane $a\alpha b'$ through the point $[r, r']$, parallel to the line $[cd, c'd']$. Through s, s' draw st, $s't'$ parallel to rp, $r'p'$, meeting cd, $c'd'$ in t, t': st, $s't'$ are the projections of the straight line which is perpendicular to the plane $a\alpha b'$, and to the two given lines $[ab, a'b']$ $[cd, c'd']$, and which is the shortest distance between these lines. Finally, constructing as in Problem XV., $s'u'$ is the true length of this line.

SOLUTION OF THE SEVERAL CASES OF THE TRIHEDRAL ANGLE.

In a trihedral angle there are six magnitudes, three faces and three inclinations, that is, three plane angles and three dihedral angles. Any three of these being given, the others may be determined. The data admitting only of six different combinations, there are only six cases for solution; viz. when there are given:

1. The three faces; or the three plane angles forming the trihedral angle.

2. Two faces and their inclination to each other; or two of the plane angles and the dihedral angle formed by their planes.
3. Two of the faces and the inclination of the third face to one of them; or two of the plane angles and the dihedral angle opposite to the plane of one of them.
4. The three inclinations; or the three dihedral angles about the trihedral angle.
5. The inclinations of two of the faces to the third face, and that third face; or two of the dihedral angles and the plane angle in the face common to them.
6. The inclinations of two of the faces to the third face, and one of those two faces; or two of the dihedral angles and the plane angle in the face opposite to one of them.

Since the plane angles forming a trihedral angle are the supplements of the opposite dihedral angles in the supplemental trihedral angle, and its dihedral angles are the supplements of the plane angles forming that trihedral angle (Prop. 46), the last three cases may be transformed into the first three in the supplemental trihedral angle, by taking the supplements of the given faces or plane angles as the inclinations or dihedral angles in another trihedral angle, and the given inclinations or dihedral angles as the supplements of the faces or plane angles forming this trihedral angle: the supplements then of the magnitudes to be determined in the supplemental trihedral angle will be those required in the original. We have, therefore, only to consider the first three of these six cases. Their solutions depend upon the methods which have been employed in some of the immediately preceding problems, namely, considering one of the faces of the trihedral angle as the horizontal plane, in turning down upon this plane the other faces of that angle, and likewise other planes which are required in the solution.

Problem XXXVII.

Given the three faces or plane angles of a trihedral angle, to find the inclinations of these faces, or the dihedral angles formed by them.

Taking the plane of one of the faces as the horizontal plane, in that plane make the angle *asb* (fig. XXXVII.) equal to the given plane angle of that face, and conceiving the other faces to be brought down upon this plane, by turning about *sa* and *sb*, let *asc* and *bsc'* be the given angles thus turned down. It is evident that the trihedral angle would be reconstructed by turning the face *asc* about the edge *sa*, and the face *bsc'* about the edge *sb* until the two sides *sc*, *sc'* coincide and form the third edge of that angle. In *sc* and *sc'* take *sf* and *sf'* equal to each other, and draw *fgh* and *f'g'h* perpendicular to *sa* and *sb* meeting each other in *h*. During the rotation of the faces *asc*, *bsc'*, the points *f* and *f'* will describe circles about the

centres g, g', in vertical planes of which the traces in the fixed plane *asb* are *gh* and $g'h$; and when *sc* and *sc'* again coincide, these points will coincide in a point in space. The lines *fg* and $f'g'$ will then form with *gh* and $g'h$ angles which are the inclinations of the lateral faces upon the horizontal face *asb* (Def. 7).

In order to determine these inclinations, it is only necessary to turn down the vertical planes described by *fg*, *fg'*, upon the horizontal plane. Turning the first about *gh*, the circle described by *fg* will be turned down upon a circle *fk* described from the centre *g* at the distance *gf*. The point *h* being common to the two vertical planes, their intersection will be a vertical line drawn from *h*; and this will be turned down upon *hk*, perpendicular to *hg*. The point F, in which the points f, f' unite in space, must be upon this intersection, and must therefore be turned down upon the point *k*, where the circle *fk* is intersected by the perpendicular *hk*. Joining therefore *gk*, the angle *kgh* will be the inclination of the faces *asb*, *asc*. If the vertical plane described by $f'g'$ be made to turn about $g'h$, we find, by a similar construction, the angle $k'g'h$ which is equal to the inclination of the faces *bsc'*, *asb*.

The inclination of the faces *asc*, *bsc'* may be found in a similar manner, by taking one of these planes as the fixed plane, and turning the other down upon this plane; but it may be more conveniently obtained in the following manner. Conceive a plane perpendicular to the third edge in which *sc* and *sc'* unite, to pass through the point F in that edge; it will cut the lateral faces in two straight lines which will contain an angle equal to the inclination sought (Defs. 3, 7). One of these on the face *asc*, turned down, is *fp* perpendicular to *sc*; and the other upon the face *bsc'*, turned down, is *fq* perpendicular to *sc'*. The points *p* and *q* where these lines meet the edges *sa*, *sb* not having changed their position, *pq* will be the trace upon the plane *asb*, of the plane containing the angles sought; and this angle will be the angle opposite to the side *pq* in a triangle of which the other two sides are *pf* and qf'. This triangle *p*F*q* brought down upon the plane *asb* by turning about *pq* will have its vertex in *m*, the intersection of circles described from the centres *p* and *q*, at the distances *pf*, *qf*; and joining *pm*, *qm*, the angle *pmq* will be equal to the inclination of the two faces *asc*, *bsc'* of the trihedral angle.

Remark.—In this construction we have to notice several verifications.

1. The lines *hk*, *hk'*, being the same vertical line *h*F turned down upon the plane *asb*, ought to be equal.

2. Since *pq* is the intersection of the planes *psq*, *p*F*q* which are each perpendicular to the projecting plane of the third edge upon the plane *asb*, it is perpendicular to that projecting plane (Prop. 34), and therefore to *shn*, the projection of the third edge, meeting it in

that plane (Def. 3). And since pq is perpendicular to the projecting plane sFn, it is perpendicular to Fn, the line drawn from F, in the plane pFq, perpendicular to the third edge; consequently when Fn is turned down upon the plane asb, the point F being at m, mn will be perpendicular to pq and be in the same straight line as sn; sh is therefore perpendicular to pq, and being produced passes through m.

3. Producing gh to i, and joining if', ik, these lines are equal, since they are the same line iF brought down, in two different ways, upon the horizontal plane; the first by the triangle iFg', turning about is, and being brought down as $if'g'$; the second, by the triangle iFh, turning about gi, and being brought down as ikh. Similarly, producing $g'h$ to i', and joining $i'f$, $i'k'$, these lines are equal.

Problem XXXVIII.

Given two faces of a trihedral angle, and their inclination to each other; or two of the plane angles and the dihedral angle contained by their planes; to find the third face and its inclinations to the given faces.

Let asb, asc (fig. XXXVIII.) be the two given faces brought down upon the same plane, by turning about as. If fgi be drawn perpendicular to sa, fg and gi will be the intersections of the faces asc, asb with the plane passing through f or F, perpendicular to sa; and the plane asc being in its true position, gf and gi will contain an angle equal to the given inclination. Making then the angle igk equal to the given inclination, if the plane of the angle contained by gf and gi turn about gi, so that this angle be brought down upon the plane asb, it will coincide with igk. Making gk equal to gf, the point f or F will be brought down on k; and the perpendicular from F on the plane asb will coincide with kh perpendicular to gi. From h, the foot of this perpendicular, drawing $hg'f'$ perpendicular to sb; describing a circle from the centre s at the distance sf, to intersect $g'f'$ in f'; and drawing $sf'c'$; bsc' will be the third face of the trihedral angle. For this face being in its true position, h is the foot of the perpendicular to the plane asb, from the point F in the edge which is the intersection of the face asc with the third face; and this face being brought down upon the plane of asb, by turning about sb, the point F will describe a circle in a plane perpendicular to sb, and will be in the line $g'f'$, at a distance from s equal to sF or sf.

The third face bsc' being found, its inclinations to the other faces are found by the last problem.

Remark.—Joining ik, if', we must have ik equal to if'.

PROBLEM XXXIX.

Given two faces of a trihedral angle, and the inclination of the third face to one of them, to find the third face and the other inclinations.

Let *asb*, *asc* (fig. XXXVII.) be the two given faces, turned down upon the same plane, and let the inclination of the third face to *asb* be given. Conceive the faces of the trihedral angle in their true position. Let a vertical plane pass through any point *g* in *sa*, perpendicular to *sa*, intersecting *asb* in *gi*, perpendicular to *sa*, and *asc* in a line also perpendicular to *sa*, and which by the revolution of the face *asc* about *sa* will describe a circle of which the radius is *gf*. Let another vertical plane also pass through *g* perpendicular to *sb*, intersecting *asb* in *go* perpendicular to *sb*, and the third face in a line also perpendicular to *sb*; these perpendiculars to *sb* will contain the given angle of inclination of the third face to the face *asb*. These two vertical planes passing through *g* will intersect in a vertical line (Prop. 34), which with *go* and the other perpendicular to *sb* will form a right-angled triangle, having the angle at *o* equal to the given angle of inclination. If this triangle be brought down upon the plane *asb*, by turning about *og*, it will be determined by making the angle *gom* equal to the given inclination, and drawing *gm* perpendicular to *go*; so that *m* is the point of intersection of the plane of the third face with the vertical line passing through *g*, brought down upon the plane *asb*. If the other vertical plane, which passes through *gi*, be brought down upon the plane *asb*, by turning about *gi*, the above vertical line passing through *g* will be brought down upon *ga*, and its intersection with the plane of the third face will be in *n*, making *gn* equal to *gm*; the line joining this intersection and *i*, brought down upon *asb*, will be *ni*; and the circle described in the vertical plane passing through *gi*, by the point *f* in the edge *sc*, brought down upon *asb*, will be the circle *qpf* whose centre is *g*. The points *n* and *i* being in the plane of the third face, and the circle *qpf* being described by a point in the edge *sc*, the intersections *p*, *q*, of *ni* with the circle, are points common to the face *asc* and the third face. The plane *fni* being replaced in its true position, at right angles to *asb*, either of the lines joining *s* and *p*, or *s* and *q*, in space, will be the third edge of a trihedral angle of which two faces are *asb* and *asc*, and the inclination of the third face to *asb* is the angle *mog*. Describing therefore circles from *i* as a centre, at the distances *ip*, *iq*, intersecting in p', q' the circle described at the distance *sf* from the centre *s*; and joining sp', sq'; either of the angles bsp' or bsq' will be the required third face, brought down upon the plane of *asb*, by turning about *sb*.

Remarks.—1. When the circle *fpq* cuts the line *ni* in two points *p*, *q* it appears that there are two solutions to the problem. When

it cuts *ni* in one point, and *ni* produced in another, the point on the other side of *sb* furnishes no solution. When the circle touches the line *ni*, the two solutions are reduced to one. Finally, when the circle and the line *ni* do not meet, the problem is impossible.

2. If from the points p and q perpendiculars be let fall upon *gi*, and from their intersections with *gi*, perpendiculars be drawn to *sb*, these lines will respectively pass through p' and q'; because p and q being in their true positions when brought down as p' and q' in the plane *asb*, by the planes in which they are turning about *sb*, these points will move in vertical planes perpendicular to *sb*.

HORIZONTAL PROJECTION.

1. In this projection, which has already been referred to (Des. Geo. 3), the positions of different points in a body are determined by their projections on a horizontal plane, and their distances from that plane. It is specially adapted to military drawing of a country and to the general representation of military works, because in these, heights and horizontal distances are commonly given numerically, and when sought are frequently to be so determined; and this projection allows us to call in arithmetic to the aid of geometry. Descriptive geometry is better adapted to particular details of military works.

2. A horizontal plane being assumed as that to which all points are referred by perpendiculars from them to the plane, the height of a point above that plane is called the *Index of Height* or *Index of Level* of that point, or sometimes simply its *Index*.

The position, therefore, of a point is determined by its projection on the horizontal plane, and its index of level; and in order to express this position in a plan, the number denoting this index is placed on one side of the projection of the point. Thus the point 13·0 (Plate IX. fig. 1) denotes a point whose index of height is 13·0, or which is at the height 13·0 above the zero horizon plane, its position on the plan showing its horizontal projection. When a point is to be referred to in a plan by a letter, the letter is placed on one side of the point, and the index of height may be placed either on the other side of the point or below the letter: thus the point a.13·0 or a_{13} would refer to the point whose horizontal projection or position on the plan is marked by the letter a, and its height above the zero horizontal plane is 13. For greater distinctness, the letter and its index of height may be inclosed in brackets: thus the point [a.13] or [a_{13}].

In naming points, the capital letters A, B, C, &c. will designate

the points themselves in space; the small letters a, b, c, &c. their horizontal projections or positions on the plan; and the Greek letters α, β, γ, &c. their indices when their numerical values are not stated, so that the Greek letters will always represent numbers.

3. To avoid negative indices to points on the plan, it is necessary that the horizontal plane assumed as the zero of level should be either above, or else below every point represented on it. By assuming the zero level below every point to be represented on the plan, the vertical co-ordinates are measured from below upwards; the indices are those of height; and we adhere to the method adopted in maps and topographical plans, where the prominent points are referred to the sea level: whereas, by assuming the zero level above the highest point, the vertical co-ordinates are measured downwards; the indices are those of depression; and our assumption is not only opposed to the principle adopted in topographical plans, but to that always employed in Descriptive Geometry; a serious objection as regards the connexion of these methods. For these reasons, in all that follows, the zero of level is assumed below the lowest point in the plan, and the indices are always those of height*.

4. The position of a straight line in space is determined when its horizontal projection and the vertical ordinates to two points in it are known; and this position is denoted in the plan by the projection of the line, and by annexing the indices to two points in it.

A straight line in space is designated in a manner similar to that adopted for a point: thus the straight line $[a.\alpha, b.\beta]$, or $[a_\alpha, b_\beta]$, or simply $a_\alpha b_\beta$, denotes the straight line joining the points $[a.\alpha]$ and $[b.\beta]$, or a_α and b_β in space (fig. 2); or the straight line joining the points A and B in space, whose horizontal projections are the points a and b in a plan, and whose indices of height are α and β respectively.

5. The position of a curve in space is determined when its horizontal and vertical projections are known, or instead of the latter when the vertical ordinates or indices to every point in it are known, but this position cannot be represented by the horizontal projection and indexed points, because the indices can only be annexed to points at certain intervals. However, the curve may be considered as known when these points are so near that it may be traced by means of them without sensible error, should this be required.

6. When the curve is to be contained in a plane given in position, its horizontal projection with the position of the plane will be sufficient to determine it completely. If all the points of the curve be on the same level, a single index and the horizontal projection are sufficient to determine its position.

* Should cases occur in the construction of works where it may be more convenient to assume the zero level above the highest point, no difficulty can present itself to the engineer who well understands the principles of the method, in making the changes which this assumption will require.

7. Since a plane is completely determined when the position of three points in it, not in a straight line, are given, the horizontal projections of three points with their indices are sufficient to determine the position of the plane in space. In this manner the position of a plane might be represented by a triangle on the plane of projection, having annexed to each angular point its index of height. This mode of representation, however, is ill-adapted to the determination of any number of points or of lines in the plane; but if the horizontal trace of the plane with the projection and index of any point in the plane be given, this object will be more readily attained, for by drawing from the projection of the given point a perpendicular on the trace, this, with the given index of the point, will determine the plane's inclination. The same object will be attained by having given the projection and index of any horizontal line in the plane and those of a point in the plane, or the projection and index of each of two horizontal lines in the plane. A plane may, on this principle, be very conveniently and usefully represented by the projections of two horizontal lines in it with their indices annexed, and the common perpendicular to them, as in figure 3.

8. If in the projection of a perpendicular to horizontal lines in a plane, points be taken at equal distances from each other, the indices of these points will be in arithmetic progression. For let the plane, and also a horizontal plane intersecting it, be cut by a vertical plane, perpendicular to its horizontal trace; let A_0A_n and A_0a_n (fig. 4) be the sections of these planes, the plane of the paper being in this case supposed vertical; then A_0A_n, being perpendicular to the plane's horizontal trace, will be a line of greatest inclination in the plane, and A_0a_n will be the horizontal projection of A_0A_n. Taking then a_1a_2, a_2a_3, a_3a_4, &c. equal to each other, drawing a_1A_1, a_2A_2, a_3A_3, a_4A_4, &c. perpendicular to A_0a_n, and A_1C_1, A_2C_2, A_3C_3, &c. parallel to it, A_2C_1, A_3C_2, A_4C_3, &c., which are the differences of the indices to the points A_1, A_2, A_3, A_4, &c., are all equal.

9. When the distances a_1a_2, a_2a_3, a_3a_4, &c. are so taken that the differences A_2C_1, A_3C_2, A_4C_3, &c. of the indices to the points A_1, A_2, A_3, A_4, &c. are the unit of length, then the ratio of A_2C_1 to A_1C_1 or a_1a_2; of A_3C_2 to a_2a_3; &c. is a measure of the plane's inclination; and the line a_1a_n (the projection of a line of greatest inclination on the plane) having the indices of the equidistant points a_1, a_2, a_3, a_4, &c. marked on it, is called the *Scale of Slope* of the plane.

10. We may thus represent a plane by a single straight line drawn in the direction of the greatest inclination, that is, at right angles to the horizontal straight lines in it, and marking, in this line, the points the differences of whose indices is the unit of length. To distinguish, however, the representation of a plane from that of a line, it is customary to draw two parallel lines at right angles to the horizontal lines in the plane, with divisions on them corresponding to

points whose indices differ by an unit. Thus the double line a_{10}, a_{16} (fig. 5) represents a plane the greatest slope on which is in the direction of this line, and in which, therefore, the horizontal trace and all horizontal lines are perpendicular to the line a_{10}, a_{16}.

It is to be borne in mind that the vertical ordinate or index of height to the point a_{10} is 10·0, and that to the point a_{16} is 16·0, as marked in the scale, that is, the line a_{10}, a_{16} being drawn in a plan which is constructed to a certain scale of feet, yards, or other unit of length, the height of the point a_{10} is 10 such units above the horizontal plane taken as the zero plane, and the point a_{16} is 16 such units above the same plane. So that dividing the line a_{10}, a_{16} into 6 equal parts in the points 11·0, 12·0, 13·0, 14·0, 15·0, each of these points is one of the units higher above the zero plane than the preceding. The distances between these points as measured on the scale of the plan give the degree of slope of the plane represented by a_{10}, a_{16}. Thus if each of the intervals a.10, 11·0; 11·0, 12·0; 12·0, 13·0; &c. be 2 on the scale of the plan, the slope of the plane will be 1 in 2; if each interval be 3, the slope will be 1 in 3; if each interval be 5·7, the slope will be 1 in 5·7, or 10 in 57; and so on.

It is necessary to distinguish clearly between the scale of a plan and the scale of slope of a plane represented on it. The intervals between the divisions on the scale of slope are always to be measured on the scale of the plan, and the numbers attached to these divisions simply indicate the heights of the points in the plane, of which the points of division are the projections.

The angle of inclination of a plane may be deduced immediately from its scale of slope, since the ratio which gives the intervals in the scale of slope is in fact the tangent of the angle of inclination.

We should observe here that a plane may be represented by its scale of slope in any part of a drawing where it may be most convenient to place it, or where it may best serve to indicate the particular plane to be represented by it.

11. When horizontal lines, or, as we may frequently term them, *horizontals*, in the plane are required to be represented on the plan, this is of course done by drawing them through the required points perpendicular to the scale of slope, as in figure 6. We may remark with reference to the scale here and in the last figure, that if these scales occurred, as they are represented, in the same plan, they would denote planes inclined in opposite directions, the one being a descending, the other an ascending plane; and the interval between two indices in fig. 5 being two-thirds of the interval in fig. 6, the slope, as measured by the differences of level in a given horizontal interval, is in the former plane one and a half times as great as in the latter.

12. There are two cases in which the scale of slope cannot be applied: when the plane is horizontal; and when it is vertical. In

the first case, a single index fixes the plane's position; and it may be conveniently represented by two straight lines making any angle with each other, and having the common index at the angular point and at their further extremities. In the second case, the trace of the plane indicates its position, and we have no other means of representing it on a plan.

13. The scales of slope of two planes represented on a plan give immediately the connexion of the planes; for the perpendiculars to the scales being the projections of horizontal lines in the planes, the intersections of those which are on the same level, or which have the same index, must meet in the straight line which is the projection of the intersection of the two planes. Thus let *ab, cd* (fig. 7) be the scales of slope of two planes: produce *ab, cd,* and extend the scales so as to have the points with corresponding indices in each; the perpendiculars to the scales drawn through these points will be the horizontals in the two planes, and each pair will meet in the straight line *pq,* which is therefore the projection of the intersection of the planes. The divisions on the line *pq,* with their indices, give the scale of slope of the line of intersection of the two planes.

It is to be understood that the angles 444, 555, 666, &c. are the projections on the plan, of the intersections of the two planes, with horizontal planes at the heights 4, 5, 6, &c. from the zero plane.

14. If instead of two planes there be several, their connexion will be represented in the plan by the projections of horizontal lines in the planes at equal intervals of height, or the projections of the intersections of the planes with horizontal planes at equal intervals from each other; and the intersections of the corresponding projections will be the projections of the intersections of the respective planes. The straight lines at right angles to the projections of the horizontal lines in the planes will give the scales of slope in the respective planes. Thus fig. 8 represents on a plan the connexion of several planes with their intersections and scales of slope, by means of the projections of horizontal lines in them at equal intervals of height.

15. As nothing limits the number of planes that may be thus represented, we may conceive their number to be so increased, both laterally and upwards or downwards, as approximately to coincide with any curved surface, and thus to represent that surface, whether regular, as a geometric surface, or irregular, as an undulating or hilly country. Such a representation of a surface would, however, only be complicated by the introduction of the intersections of the approximating planes. Without these the surface would be represented simply by the projections of the intersections of its approximating planes with equidistant horizontal planes. Thus supposing the planes in fig. 8 to touch a curved surface so as to form a first approximation to it, the projections of the level lines without the

intersections will be an approximate representation of the surface itself. It is evident that the greater the number of touching planes, the more nearly will they coincide with the curved surface, and the more correctly will the projection of their intersections with the equidistant horizontal planes represent the surface on the plan. The projections therefore of the intersections of the surface itself with the equidistant horizontal planes will best represent that surface, dispensing with the touching planes, except where it may be necessary to determine the scale of slope on a limited portion of the surface. As the scale of slope on a plane is a perpendicular to the projection of a horizontal straight line in the plane, the scale of slope at any point on a surface is a perpendicular at that point to the projection of the horizontal line passing through the point. Figure 9 thus represents a curved surface to which the six planes represented in figure 8 may be considered an approximation; and figure 10 represents a more complex surface having flexures in various directions, with some of the lines of greatest slope in directions approximately perpendicular to the projections of the curved intersections of the surface with the equidistant horizontal planes. Such a plan may represent a portion of country, without other details of the surface being shown on it, and immediately presents an idea of the general nature of that surface as regards its slopes, the greater or less steepness of which is shown by the approach of the horizontal curves or their receding from each other.

16. The horizontal curves which thus represent a country are called the horizontal *Contours* of the country, and a plan or map in which these contours are represented is called a *Contoured Plan* or *Map*. The advantages of such plans and maps, whether for the purposes of military or civil engineering, are incalculable. They at once speak to the eye, conveying a knowledge of the most important feature of the country as regards the operations in both those branches of service, and, besides, they furnish data for the details of these operations. It is not here a question how materials are to be obtained for the construction of such plans, but we may remark that they must be furnished by the operations of leveling, whether by actually tracing the levels at the required intervals of height, or by any other mode which may be more convenient.

17. The principles of this method of representation as applied to points, lines, whether straight or curved, and surfaces, whether plane or curved, having been explained, we proceed to some problems which most commonly occur in the application of the method.

Problem I.

A straight line AB *being given by its projection* ab, *and the indices* α, β *of two points in it whose projections are* a *and* b :—

1. *To find the index of another point* C *in it whose projection is* c ;
2. *To find the projection of a point in it whose index* γ *is given.*

1. It is evident that if, in the vertical projecting plane of AB, a straight line be drawn through B, supposed to be the lower point, parallel to the horizontal projection *ab*, this line, with the vertical projecting lines of A and C, will form two similar right-angled triangles of which the bases are equal to *ba* and *bc*, and the perpendiculars to the excess of A's index above B's, and the excess of C's above B's.

Geometrically, the excess of C's index above B's, is immediately obtained by constructing these right-angled triangles according to the scale of the plan.

Arithmetically, C's index is thus obtained.

α, β being the indices of A and B, let γ be that of C; then by the similar triangles

$$\frac{\gamma-\beta}{\alpha-\beta}=\frac{bc}{ba};\ \therefore \gamma=\beta+(\alpha-\beta)\frac{bc}{ba},\ \text{or}\ \frac{\alpha.bc+\beta.ac}{ba}.$$

2. When the index γ of the point C is given, to find its projection, we have—

Geometrically, the distance *bc* is the base of a right-angled triangle of which the perpendicular is $\gamma-\beta$, the excess of the given index of C above that of B, and which is similar to the right-angled triangle of which the base is *ab*, and perpendicular is $\alpha-\beta$, the excess of A's index above B's.

Arithmetically, we have by similar triangles—

$$\frac{bc}{ba}=\frac{\gamma-\beta}{\alpha-\beta};\ \therefore\ bc=\frac{\gamma-\beta}{\alpha-\beta}.ba.$$

In the solution of the problem, the point C has been assumed between A and B, but when the point is beyond A or B, the solution is evidently obtained on the same principle.

When the straight line AB is horizontal, the index of every point in it is the same, and consequently when the projection of the point C is given, the point is given, since its index is so; but the projection of a point in the line cannot be determined from its index.

When AB is vertical, then the projection of every point in it being the same point, any point in the line is only known by its index.

PROBLEM II.

A straight line AB *being given by its projection and the indices of the points* A, B, *to find its true length and its inclination to the horizon.*

The true length of the line AB is evidently the hypothenuse of a right-angled triangle of which the base is the projection of AB, and its perpendicular is the difference of the indices of A and B; whence it is immediately found, either Geometrically or Arithmetically, whichever may be most convenient.

The inclination of the straight line to the horizontal plane is the angle which it makes with its projection, and this angle is the angle at the base of the above right-angled triangle.

If the slope of the straight line, as that of a plane expressed in its scale of slope, be required, this slope is expressed by a fraction whose numerator is the difference of the indices of A and B, and the denominator is the projection of AB.

If the converse of the former part of this problem were proposed, namely, *a true length in a straight line, given by its projection and the indices of two points in it, being given, to find its projection*; or, which amounts to the same, *from a given indefinite straight line to cut off a given length*; we should then have to find the base of a right-angled triangle similar to that which determines the scale of slope of the line, and the hypothenuse of which is the given length to be cut off. Arithmetically, we should have to find a fourth proportional to, the true distance between two points whose projections are given, the distance between the projections of those points, and the given length to be cut off.

PROBLEM III.

Through a given point in a given vertical plane, to draw a straight line that shall have a given inclination to the horizon.

The projection of the required straight line will coincide with the straight line which is the projection of the given vertical plane; and the projection of the given point being in the same straight line, if another point in it be assumed as the projection of a second point through which the line passes, its index will readily be found in the following manner.

Let *mn* (fig. 11) be the projection of the given vertical plane, *a* the projection of the given point A, and BAC the given angle. In *mn* take another point *b*; make AC equal to *ab*, and draw CB perpendicular to AC. Let the index of *a* equal α, that of *b* equal β, and suppose BC equal γ; then if $\beta = \alpha + \gamma$, the straight line passing through the points A and B whose projections are *a* and *b*, and indices α and β will make the given angle BAC with the horizon.

PROBLEM IV.

To determine the intersection of two given straight lines AB *and* CD *in space, in the case of their meeting: that is,* $a_\alpha\ b_\beta$ *and* $c_\gamma d_\delta$ *being two given straight lines, it is required to find the projection and the index of their point of intersection, in the case of their meeting.*

The projection of the point of intersection of the two straight lines being the point of intersection e of their projections ab, cd (fig. 12), it is only necessary to determine (Prob. I.) the index of the point e in either of the lines $a_\alpha b_\beta$ or $c_\gamma d_\delta$.

Thus ε being the index of the point of intersection, we have

$$\varepsilon=\frac{\alpha.be+\beta.ae}{ab}, \quad \text{or } \varepsilon=\frac{\gamma.de+\delta.ce}{cd}.$$

If the lines intersect, it is evident that the index of the point e must be the same by whichever of the two given straight lines it is determined.

When the two given straight lines are in the same vertical plane, their projections coincide, and they necessarily intersect, except in the case of their being parallel. In order to determine the projection and index of the point of intersection, it is only necessary to draw the straight lines as they exist in the vertical plane.

Let AB, CD (fig. 13) be the two given straight lines, of which the projections are ab and cd, and the indices of their extremities are $Aa=\alpha$, $Bb=\beta$, $Cc=\gamma$, $Dd=\delta$; then the projection and index of their intersection is thus found:

Geometrically. By measuring ae and eE on the scale of the plan, the values of these lines will be obtained to the degree of accuracy generally required.

Arithmetically. Drawing AF and DG parallel to ab, we have

$$\frac{\text{AH}}{\text{HE}}=\frac{\text{AF}}{\text{FB}}, \text{ and } \frac{\text{DI}}{\text{IE}}=\frac{\text{DG}}{\text{GC}}.$$

Putting $ab=b$, $ac=c$, $ad=d$, $\text{AH}=ae=x$, and $\text{HE}=y$, these proportions become

$$\frac{x}{y}=\frac{b}{\beta-\alpha}, \text{ and } \frac{d-x}{y-(\delta-\alpha)}=\frac{d-c}{\gamma-\delta}.$$

From these equations we obtain

$$x=\frac{(\gamma-\delta)d+(\delta-\alpha)(d-c)}{(\gamma-\delta)b+(\beta-\alpha)(d-c)}.b;\ y=\frac{(\gamma-\delta)d+(\delta-\alpha)(d-c)}{(\gamma-\delta)b+(\beta-\alpha)(d-c)}.(\beta-\alpha).$$

Substituting in these the numerical values of the letters, we have those of $ae=x$, and $\text{E}e=\alpha+y$.

When the straight lines are both horizontal, they can only intersect when they have the same index. In this case, the projection of their intersection will be the intersection of their projections, and its index will be the common index of the straight lines.

PROBLEM V.

Through a given point A *whose projection is* a, *and index* α, *to draw a straight line parallel to a given straight line* BC *whose projection is* bc (fig. 14), *and the indices of two points* B, C *in it are* β, γ.

The projection of the line required will be parallel to that of the given line (Props. 35, 27); therefore through *a* draw *ad* parallel to *bc*, as the projection of the required straight line. Join *ba*, and through *c* draw *cd* parallel to *ba*; if D be the point whose projection is *d*, and whose index δ is $\alpha+\gamma-\beta$, the straight line AD in space will be parallel to the given straight line BC. For AD and BC are in parallel vertical planes, and they make equal angles with horizontal lines in those planes, because *ad* is equal to *bc*, and $\delta-\alpha$ is equal to $\gamma-\beta$.

PROBLEM VI.

A plane being known by sufficient data, to determine its scale of slope.

Case 1.—*The plane being given by three points in it, not in a straight line.*

Let a_{11}, b_{47}, c_{32} (fig. 15) be the projections of the three points A, B, C in the plane, with their indices. Join *ab*, and in it find the point *d* which is the projection of the point in AB, having the same index 32·0 as the third point C (Prob. I.); and join *cd*, which is therefore the projection of a horizontal straight line in the plane. Since *cd* is the projection of a horizontal in the plane, drawing *af* parallel to *cd*, it is also the projection of a horizontal in the plane. Drawing any line *ef* perpendicular to *cd*, this will be the projection of a line in the plane in the direction of the greatest inclination, or in the direction of the scale of slope (9, 10), and *e*, *f* will be the projections of points in the plane at the heights 32 and 11 from the zero plane. Dividing, therefore, the distance *fe* into 21 equal parts, the several points will be the projections of points in the plane at intervals of an unit in level: and thus the scale of slope of the given plane is constructed.

Case 2.—*The plane being given by two points in it, and its angle of inclination to the horizon*; or, which is the same problem, *to determine the scale of slope of a plane that shall pass through two given points, and have a given inclination to the horizon.*

Let the given points in the plane be a_{35}, b_{20} (fig. 16), and LMN the given angle of inclination of the plane, which must not be less than the inclination of the straight line $a_{35}b_{20}$ to the horizon. Draw MP perpendicular to MN, and equal to fifteen on the scale of the plan, the difference of the indices of the points A and B; draw PL parallel to MN, and LN to MP. From the centre *a* at the distance MN describe a circle; and from *b* draw *bc* touching it in *c*. The

plane $a_{20}b_{20}c_{20}$ is a horizontal plane at the height 20 from the zero plane, and the point A, (a_{35}) being at the height 15, equal to LN, above this plane, any line joining A and the circumference of the circle will make an angle with the horizontal plane equal to the given angle LMN. The line joining A and c_{20}, or a_{35} and c_{20}, and the horizontal line $a_{20}c_{20}$ are both perpendicular to the horizontal line $b_{20}c_{20}$, and consequently the angle $a_{35}c_{20}a_{20}$ is the angle of inclination of the plane $a_{35}b_{20}c_{20}$ to the horizon, which is therefore equal to the given angle LMN. Dividing the horizontal line *ca* into 15 equal parts, we have the required scale of slope.

Case 3.—*The plane being given by a point in it, its inclination to the horizon and the direction of inclination.*

The inclination of the plane being given, to find its scale we have only to draw a right-angled triangle, having the angle at the base equal to the given inclination, and the perpendicular opposite equal to any number of units; the base being divided into the same number of units will give the divisions of the scale. Drawing, therefore, through the given point, a straight line in the given direction, and dividing it from that point at intervals equal to those in the base of the right-angled triangle, it will be the required scale of the plane.

Problem VII.

The scale of slope of a plane being given, to find the index of a point in the plane when its projection is given.

A straight line drawn through the projection of the given point, perpendicular to the given scale, will be the projection of a horizontal line passing through the given point in the plane; and therefore its intersection with the scale will give the index of the point.

The converse of this problem, viz. *to find the projection of a point in the plane when its index is given,* is indeterminate, because every point in the horizontal line which is in the plane, and passes through the point in the scale denoted by the given index, has the same index.

Problem VIII.

On a given plane, to draw a straight line to pass through a given point and to have a given inclination to the horizon, not greater than that of the given plane.

From the given inclination, the ratio of the base to the perpendicular, in a right-angled triangle of which the hypothenuse has that inclination to the base, is given. Draw therefore on the given plane, whose scale of slope is 38, 41 (Plate X. fig. 17), a horizontal line having an index differing by one, or any number of units from that of the

given point [38·7], and from the projection of that point as a centre, at the distance of the base of the corresponding right-angled triangle, describe a circle, cutting the projection of the horizontal line [39·7, 39·7] : the straight line [38·7, 39·7] joining the projection of the given point and either of the intersections will be the projection of a straight line in the plane, and which has the given inclination.

Problem IX.

To find the intersection of two given planes.

The general solution of this problem has already been given (13); but we have now to make some remarks on the character of the dihedral angle formed by the surfaces of the planes, and also to notice some particular cases of the problem.

The intersections of pairs of horizontals, having the same index in each of the planes, determine the line of intersection of the planes (13). When the angle contained by the pair of horizontals of the *higher* level *is included by* the angle contained by the pair of the *lower* level, or the angular point of the *higher* is turned *towards* that of the *lower*, the angle formed by the surfaces is *salient*, or they form a *ridge*: when the angle contained by the horizontals of the *higher* level *includes* the angle contained by those of the *lower*, or the angular point of the *higher* is turned *from* that of the *lower*, the angle formed by the surfaces is *re-entering*, or they form a *channel*. Or when the *higher* divisions of the scales of the two planes *approximate*, the planes form a *ridge*; when the *lower* divisions *approximate*, the planes form a *channel*.

Particular cases.—1. *When one of the planes is horizontal*, we have only to find on the scale of the inclined plane a point which has the same index as the horizontal plane: the horizontal straight line passing through this point will be the projection of the planes' intersection.

2. *When the horizontal straight lines in the two planes are parallel*, the intersection of the planes will be horizontal and parallel to these; and its projection must be a perpendicular to both scales, and pass through points in each which have the same index. By placing the scales of the two planes contiguous to each other, the coincidence of the points which have the same index may be immediately seen, at least approximately, and the intersection thus determined.

The intersection of the two planes may, however, be readily determined by finding their intersections with a third auxiliary plane, since the point of meeting of the two intersections must be a point in the intersection of the two planes; and the intersection will be the perpendicular from this point on either of the scales.

Let *ab*, *cd*, as divided (fig. 18), be the scales of the two planes.

It is evident from inspection that the index of the horizontal line in the two planes which has a common index must be nearly 28·5.

To find this index by construction, through any two points, as 24 and 27 in each of the scales, draw perpendiculars to the scales; these will be horizontals at the respective heights in the two planes. Draw two parallels *ef, gh,* to represent the horizontals at the heights 24 and 27 in an auxiliary plane, intersecting the corresponding horizontals in the two planes in *f, i* and *h, k*; join *fh, ik,* and produce them to meet in *l*: *fl* is the intersection of the auxiliary plane with the plane whose scale is *ab,* and *il* its intersection with the plane whose scale is *cd*; and consequently *l* the intersection of *fl* and *il,* is a point in the intersection of the two planes; drawing therefore the horizontal *lm,* perpendicular to both *ab* and *cd,* it is the intersection of the planes of which *ab* and *cd* are the scales.

When the construction is correct, *lm* will cut the two scales in points having the same index.

Problem X.

To find the intersection of a given straight line with a given plane.

Let $b_{27\cdot4}a_{30\cdot7}$ (fig. 19) be the given straight line, and $c_{27}d_{31}$ the scale of slope of the given plane. Through $a_{30\cdot7}$ and $b_{27\cdot4}$ draw the parallels *ae, bf* as the horizontals in a plane passing through the given line; and through the points 30·7, 27·4 of the given scale, draw the horizontals in the given plane, intersecting the corresponding horizontals in the auxiliary plane in the points $e_{30\cdot7}$, $f_{27\cdot4}$; the line joining these points is the intersection of the auxiliary plane with the given plane. The point $g_{29\cdot3}$, in which the given line cuts this intersection, is therefore the intersection of that line with the given plane.

Problem XI.

Through a given point, to draw a plane parallel to a given plane.

This problem is immediately reduced to Case 3, Prob. VI.; for if through the given point a plane be drawn having the same inclination to the horizon as the given plane, and in the same direction, it will be parallel to that plane.

The scale, however, of the required plane may be more readily determined from the consideration, that, since the planes are parallel, the difference between the indices of points in them which have the same projection must be everywhere the same; and consequently the scales of the two planes will be the same as regards the intervals of their divisions, but commencing at different indices, as in figure 20, where 22·4 is the index of the given point, and 19, 23 is the scale of the plane passing through it, parallel to the plane whose scale is 37, 40.

Problem XII.

Through two given straight lines, to draw planes parallel to each other.

If through a point in one of the given straight lines a straight line be drawn parallel to the other, this last line will be parallel to the plane passing through the intersecting lines (Prop. 21); and will therefore be wholly contained in a plane passing through any point in it, and parallel to the former plane. On these principles we have the construction in figure 21. Through the point $a_{35 \cdot 4}$ in one of the given straight lines $a_{35 \cdot 4}b_{33 \cdot 1}$, a straight line $a_{35 \cdot 4}e_{33 \cdot 7}$ is drawn parallel to the other straight line $c_{21 \cdot 2}d_{19 \cdot 5}$ (Prob. V.), 35·4 — 33·7 being equal to 21·2 — 19·5, and the projections of the lines being equal. Finding the point f, in the straight line $a_{35 \cdot 4}b_{33 \cdot 1}$, whose index is 33·7 (Prob. I.), the straight line $f_{33 \cdot 7}e_{33 \cdot 7}$ is a horizontal in the plane of the intersecting lines; and its scale of slope being perpendicular to this, is readily constructed. The scale of the plane parallel to this plane, and passing through a point in the line $c_{21 \cdot 2}d_{19 \cdot 5}$, and therefore through the line, is determined by Problem XI.

Problem XIII.

From a given point, to let fall a perpendicular on a given plane.

The horizontal line in the plane, passing through the foot of the perpendicular, being at right angles to the perpendicular, and also to the straight line of greatest inclination in the plane, the plane passing through the perpendicular and the line of greatest inclination is at right angles to the horizontal straight line (Prop. 4), that is, it is vertical; and it is therefore the vertical projecting plane both of the perpendicular to the given plane and of a straight line of greatest inclination in it. The projection of the perpendicular, therefore, coinciding with the direction of the scale of slope of the plane, if besides the given point, the index of a second point in it be found, the perpendicular will be determined.

We will first consider the perpendicular to the given plane as it exists in the vertical plane passing through it, and cutting the given plane in a straight line of greatest inclination. Let A (fig. 22) be the given point, and BC the intersection of the vertical plane passing through the perpendicular from A to the given plane, with this plane. Let AH be a horizontal straight line cutting BC in H. Take HE equal to any number of units on the scale of the plan, and draw EF perpendicular to AH : make AG equal to EF, and draw GI perpendicular to AH. Then since the triangles AGI and FEH are equiangular, and AG is equal to FE, GI is equal to HE; that is, if AG be taken equal to the excess of the index of the point H or the point A above the index of the point F in the plane, the excess

of the index of the given point A above that of the point I will be the number of units in HE on the scale of the plan. This principle gives the following very simple method of determining the index of a second point in the perpendicular to the given plane.

Let a_{78} be the given point, 40, 80 the scale of slope of the given plane, and 0, 50 the scale of the plan. The projection of the perpendicular to the plane is *al* parallel to the scale of slope. In this scale find the point *h* whose index is 78, that of a_{78}; take *hf* equal to any number of units, 20, on the scale of the plan; and find the index of the point *f*, 64·67, by means of the scale of slope. Take *ai*, on the projection of the perpendicular, equal to 78 − 64·67 or 13·33 on the scale of the plan: then from the foregoing principle, the index of the point *i* is 78 − 20 = 58, the index of a_{78} diminished by the number of units (on the scale of the plan) in *hf*.

It is evident that it is unnecessary to find the point *h*, on the scale of slope, having the same index as the given point *a*, since the distance equal to the assumed number of units on the scale of the plan being measured from any other point in the scale of slope, the difference in the indices of the two points will be the same.

Two points being found in the perpendicular, we may find any number of points in it (Prob. I.), and construct its scale. We may also find its intersection with the given plane (Prob. X.), and the true length of this perpendicular (Prob. II.), or the distance of the given point from the plane.

When the given point is in the plane, a second point in the perpendicular is determined on precisely the same principle.

When the scale of slope of the plane is not given, it is not necessary to construct it; all that is required for the determination of a second point in the perpendicular being the projection of two horizontals in the plane. Thus let 53, 53 and 65, 65 (fig. 23) be two given horizontals in the plane, and a_{84} the given point. The projection of the perpendicular to the plane is in the direction *al* perpendicular to the horizontals. The distance between these is 18 on the scale of the plan, and *ai* being taken on that scale equal to 12 (65 − 53), the index of *i* is 84 − 18 or 66.

Problem XIV.

Through a given point, to draw a plane perpendicular to a given straight line.

The given straight line being perpendicular to the required plane, the scale of slope of this plane must be in the direction of the projection of the line or parallel to it. Drawing, therefore, for the scale of slope of the required plane, a straight line parallel to the projection of the given straight line, and passing through the projection of the given point, and taking this point as a point deter-

mined in the scale of slope, a second point in that scale may be determined, in precisely the same manner as the second point in the perpendicular was determined in the last problem.

Let $a_{57}\, b_4$ (fig. 24) be the given straight line, and c_{63} the given point. Through c draw cd parallel to ab, for the scale of slope of the required plane; 63 is therefore the index of the point c in that scale. Taking any number of units, say 10, on the scale of the plan, and applying the distance to the scale $a_{57}\, b_4$, it corresponds to 15; taking therefore 15 on the scale of the plan, and applying this distance from c_{63} to e on the scale of slope cd, it must correspond to a difference in the index 10, the number of units first taken, giving 73 for the index of the point e. Having these two points in the scale of slope of the required plane, that scale may be constructed.

Problem XV.

From a given point, to let fall a perpendicular on a given straight line.

Drawing a plane through the given point, perpendicular to the given straight line, by the last problem, its intersection with that line is to be determined by Problem X.: then the straight line drawn from the given point to the point of intersection will be perpendicular to the given straight line.

Problem XVI.

To draw a plane that shall pass through a given straight line and be perpendicular to another given plane.

If from a point in the given straight line a perpendicular be drawn to the given plane (Prob. XIII.), then the plane which passes through this perpendicular and the given straight line, being perpendicular to the given plane, will be the plane required.

Let $a_{75}\, b_{43}$ (fig. 25) be the given straight line, and 10, 40 the scale of slope of the given plane. From b_{43} draw b_{43} 23 perpendicular to the given plane (Prob. XIII.), and in it find the point 75 (Prob. I.); the straight line joining this point and a_{75} will be one of the horizontals in the required plane; and a perpendicular to this, divided by parallels to it passing through the divisions on the scale of the perpendicular to the given plane, will be the scale of slope of the plane required.

18. All the details of the operations required in the construction of the preceding problems having been fully explained, and the method of performing them illustrated by solutions of particular examples, in those which follow we shall simply explain the principles on which their solutions depend, leaving the construction of particular examples, by means of scales, to the student, as exercises of the foregoing method.

Problem XVII.

To measure the angle formed by two given straight lines which meet.

From a point in one of the lines let fall a perpendicular on the other (Prob. XV.): the angle required is the angle at the base of the right-angled triangle thus formed in space, and of which the three sides are determined. Arithmetically, the tangent of the required angle is the perpendicular thus drawn, divided by the distance between the foot of it and the point of intersection of the given lines.

A very simple method, however, of measuring the angle is, to join two given points in the given straight lines; to determine the true lengths of the three lines of which we have thus the projections (Prob. II.); to construct the triangle of which these true lengths are the sides; and in it to measure the corresponding angle. In this construction we in fact conceive a plane to pass through the given lines, and determine the angle they make, by constructing, in that plane, a triangle whose three sides are given.

Problem XVIII.

To measure the angle of inclination of a given straight line to a given plane.

From a point in the given straight line draw a perpendicular on the given plane, and join the foot of the perpendicular and the point of intersection of the given line with the plane: the angle required is the angle at the base of the right-angled triangle thus formed in space. Arithmetically, the tangent of the required angle is the perpendicular divided by the base of this right-angled triangle.

Problem XIX.

To measure the angle formed by two given planes.

Through a point in the intersection of the two planes draw a plane perpendicular to that line (Prob. XIV.): the angle formed by the intersections of this plane with the given planes (Prob. XVII.) measures the inclination of these planes.

Arithmetically. From any point between the planes let fall perpendiculars on each of them, and produce the perpendicular on one of the planes to meet the other plane: the angle which the perpendicular produced makes with the other evidently measures the angle formed by the two planes (Prop. 43); and its tangent is the line intercepted between the foot of the perpendicular on the plane and the point of intersection of the line produced with the same plane, divided by the perpendicular.

Problem XX.

Through a given point, to draw a straight line that shall make a given angle with a given straight line.

Conceive a plane to pass through the given point and the given straight line; then if in this plane a straight line be drawn from the given point to a given point in the given line, the angle formed by the two lines can be constructed (Prob. XVII.). Let A (fig. 26) be the given point in space, BC the given straight line, and ACB the angle thus formed in the plane passing through A and BC. At any point D in BC make the angle CDE equal to the given angle, and through A draw AF parallel to DE: AF makes the required angle with the given straight line BC. CF, the true distance from a given point C, in BC, of the point F, to which the straight line must be drawn from A to make the given angle with BC, being known from this construction, we have only in the given projection of BC to find the projection of the point F (Prob. II.); then the straight line joining the projection of this point and that of A will be the projection of the required straight line.

Problem XXI.

Through a given straight line, to draw a plane that shall make a given angle with a given plane.

If we conceive a plane perpendicular to the intersection of the required plane with that given, to pass through a given point in the given straight line, and a perpendicular to be drawn from that point to the given plane, this perpendicular and the intersections of the first plane, with the other two, will form a right-angled triangle, of which the angle at the base is the given angle of inclination; and consequently the ratio of the base to the perpendicular in this triangle is known. The length of the perpendicular from the given point being therefore determined, the length of the base of this right-angled triangle is also known. The given straight line being produced to meet the given plane, the point of intersection can be found (Prob. X.), and therefore the length of the line from this point to the foot of the perpendicular determined. Let C (fig. 27) be the foot of the perpendicular, drawn from a given point in the given straight line, to the given plane; D the intersection of the given line with the given plane; DG the intersection of the required plane with that given; and CE the intersection of the plane perpendicular to DG with the given plane. The length of CE is determined from its ratio to the perpendicular at C; the length of DC can also be determined (Prob. II.), the points D and C being known; and therefore also the length of DE the remaining side of

the right-angled triangle. Let CF be perpendicular to DC, meeting DG in F; then since DE : EC : : DC : CF, the lengths of DE, EC and DC being determined, that of CF becomes known.

The point D, the intersection of the given straight line with the given plane, and therefore also a point in the intersection of the required plane with it, being determined, and also the point C, the straight line DC becomes known in magnitude and position; and CF being drawn perpendicular to it, and of the length determined as above, F, another point in the intersection of the required plane with that given, is determined. The plane, therefore, which passes through the given straight line and the straight line DF, thus determined, will make the required angle with the given plane.

When the given straight line is parallel to the given plane, the intersection of the two planes will be parallel to that line (Prop. 23). A plane, therefore, drawn perpendicular to the given line through a given point in it, being perpendicular to the intersection of the two planes (Prop. 18), is perpendicular to each of them (Prop. 32); and its intersection with the given plane will be at right angles to the perpendicular let fall upon that plane from the given point (Prop. 33 and Def. 3). We have therefore only to take a point in this intersection at a distance from the foot of the perpendicular, in the required ratio to the length of that perpendicular; the straight line passing through this point, at right angles to the intersection, will be the intersection of the plane required, with the given plane.

19. In the foregoing problems, which relate to the plane and the straight line, all the graphic operations which are required in their construction may be accurately performed, but this is no longer the case when such operations are to be performed on surfaces which are not plane, and particularly when these are only known by a certain number of their horizontal contours, the bands of the surface between these remaining undetermined. It is true that we may substitute for these undetermined bands, surfaces which are connected with the bounding horizontal curves by a particular law, and which more or less approximate to the real surfaces, but the horizontal curves themselves, in the case of a country, to which we now particularly refer, are not derived from any known law, nor can they be so connected. We cannot, therefore, perform exact operations on these curves; and it is only approximately that we can find their intersections with other lines; that we can draw tangents or perpendiculars to them; and consequently that we can execute operations on the surfaces which they define. We cannot here go much into this branch of our subject, but must be satisfied with pointing out some of the leading principles in two or three of the simplest problems.

PROBLEM XXII.

A surface being given by its horizontal contours, to trace upon it, commencing at a given point, a curve of which the tangent shall, in every point, make the same angle with the horizontal plane.

If we assume, as an approximation, that the chord, joining the points where the required curve cuts two adjacent horizontal contours, is parallel to the tangent to the curve, at the middle point of this portion, then we have only to find the projections of the chords which make the required angle with the horizon. The height between the planes of the horizontal contours being known, if we take this as the perpendicular of a right-angled triangle of which the angle at the base is the given angle of inclination, the base of the triangle will be the projection of one of the required chords. Commencing then at the given point as a centre, and taking the base thus found as the radius of a circle, the intersection of this circle with the adjacent contour will be the projection of a second point in the required curve. Taking the point thus found as the centre, and with the same radius, the intersection of the circle with the next following contour will be the projection of another point in the curve. Proceeding from this point in the same manner, the projection of another point will be found; and thus will be found, in succession, the projections of the intersections of the required curve with the given contours. The lines joining the points thus determined are an approximation (the nearest that the data enable us to obtain) to the projection of the required curve; and the degree of approximation will depend upon the proximity of the horizontal planes of contour.

In fig. 28 are represented the lines, thus determined, which approximate to two curves, commencing from different points in a surface given by its horizontal contours, both curves in every part forming the same given angle with the horizon.

In order that the curve may be continuous to the extent of the given surface, or in other words, that the solution may not become impossible at any point, it is evidently necessary that at each point the base of the slope or the radius shall not be less than the least distance of the point from the next contour.

If abrupt turnings in the curve be not excluded, then at each point, in succession, we may have two points determined by the intersection of the circle with the next following contour; and we should thus obtain a great variety of curves having the required inclination, among which that combination is to be selected which is best adapted to the particular object in view.

Among the practical applications of this problem, we may especially refer to the construction of roads on the face of a slope, and that are required to have the same degree of inclination in every part. If fig. 28 be supposed to represent, by horizontal contours,

the surface of a portion of country, the lines drawn from one contour to another may be considered to represent roads which, in every part, have the same degree of inclination. Of two of these commencing at the same point, one is carried down in a succession of zigzags.

Problem XXIII.

To determine the intersection of a given plane with a surface given by its horizontal contours.

This is determined on the same principle as the intersection of two planes (Prob. IX.); the several horizontals in the plane, having the same index as the given contours of the surface, being drawn at right angles to the plane's scale of slope, their intersections with the corresponding horizontal contours of the surface will evidently give points of intersection of the plane with the contoured surface; and the line joining these points will be their line of intersection. In joining the points of intersection it is, however, to be observed, that those correspond to the same intersection of the plane with the surface, in which the horizontals of the plane, taken in the same direction, having been exterior to the surface, penetrate it, and become interior; or having been interior, pass out and become exterior; and it is the succeeding corresponding points which are to be joined, to obtain the intersections of the plane with the surface. In fig. 29 the intersections of a plane, given by its scale of slope, with a surface given by its horizontal contours, are represented by the lines joining the corresponding intersections of the dotted horizontals of the plane with contours of the surface, which have the same index.

Problem XXIV.

To find the intersection of a given straight line with a surface given by its horizontal contours.

If through the given straight line we conceive a plane to pass, of which this line is the scale of slope, and find the intersections of this plane with the given surface, the point where the projection of the given straight line meets the projection of the line of intersection of the plane and the surface will evidently be the projection of the point of intersection of the given straight line with the given surface (fig. 29).

Problem XXV.

To find the intersection of two surfaces given by their horizontal contours.

This will evidently be determined by finding the intersections of the horizontal contours of one surface with the corresponding ones of the other, and drawing a line through those points of intersection (fig. 30) in a manner precisely similar to that by which the intersections of a plane with a surface have been determined (Prob. XXIII.).

ISOMETRIC PERSPECTIVE.

1. The leading feature in descriptive geometry is that all lines drawn on the paper are represented in their true dimensions, and, when in the same plane, in their true relative positions. Although the method we are about to describe has not this advantage, and, besides, cannot, conveniently at least, be applied to the solution of problems, it has that of exhibiting a conventional picture, readily understood, and in which lines in the three principal directions are represented to the same scale. In machinery*, in astronomical and other instruments, and in buildings, the principal parts to be represented are very commonly in planes at right angles to each other, and it is therefore to such objects that the method is peculiarly applicable.

2. The principle of *Isometric Perspective* will be best understood by conceiving a cube to be so placed on the plane of the paper, that its diagonal is perpendicular to that plane. In this position, the three lower edges of the cube (and also its other edges) make equal angles with the plane of the paper, and consequently their orthographic projections on it will be equal straight lines radiating from a point; as the angles between these edges are equal, and their planes are equally inclined to the plane of projection, the angles between the projections of these edges will be equal, and therefore each of them will be a third of four right angles. The diagonal of the cube being at right angles to the plane of projection, calling an edge of the cube 1, its isometric projection is equal to the perpendicular let fall from the right angle, on the opposite side, of a right-angled triangle of which the sides are 1, $\sqrt{2}$ and $\sqrt{3}$: it is therefore equal to $\frac{1.\sqrt{2}}{\sqrt{3}}=\frac{1}{3}\sqrt{6}=\cdot8164966$.

3. ABDEFGC (Plate XI. fig. 1) represents such a projection of a cube as has been described. One diagonal of this cube being perpendicular to the plane of the paper, its orthographic projection on that plane will be a point. O denoting the projection of the lower extremity of this diagonal, and C that of its upper extremity, OD, OF, OA will represent the projections of the cube's lower edges; CB, CG, CE those of its upper edges; and AB, BD, DE, EF, FG, GA those of its other edges which here bound its picture. All these edges being equally inclined to the plane of projection, their projections must all be equal.

* The method originated, with its inventor, the late Professor Farish, in attempts to give such pictures of machinery, as should enable a person, acquainted with the principles on which the pictures were drawn, to put the several parts in their proper relative positions, without any other assistance. Cambridge Transactions, vol. i. p. 1.

4. The angle contained by any two intersecting edges is a right angle, but the angle contained by their projections cannot be the same in all cases: the projections of all right angles about the points O and C, and of all angles opposite to these, will be a third of four right angles, or 120°; and those of all others will be a third of two right angles, or 60°, since the projection of each face of the cube is a parallelogram.

5. All the faces of the cube are projected in, and represented by equal rhombuses, but the projections of the diagonals of these faces are not equal. The projections of diagonals which are parallel to the plane of projection are equal to those diagonals, that is, are each equal to $\sqrt{2}$, an edge of the cube being 1; and the projections of diagonals which are inclined to the plane of projection are equal to the projections of an edge, that is, to $\frac{1}{3}\sqrt{6}$. Thus each of the diagonals AF, FD, DA, EB, BG, GE, which are projections of diagonals parallel to the plane of projection, is equal to $\sqrt{2}$; and each of the diagonals GO, EO, BO, DC, AC, FC, which are projections of inclined diagonals, is equal to $\frac{1}{3}\sqrt{6}$.

6. If OD, OF, OA, the projections of the lower edges of the cube, be produced, then OX, OY, OZ represent similar projections of three axes at right angles to each other, with reference to which the position of any point in space may either be given or be required to be determined. Lines measured along these axes, and likewise any lines parallel to them, will all be to the same scale; so that the position of a point in space, with reference to three rectangular axes, being given in terms of any unit of length, its position will be correctly represented by measuring, by means of a scale of equal parts, from O along the respective axes or lines parallel to them, lines expressing the distances from the origin or point of the intersection of the axes. Thus let the distances along the axes OX, OY, OZ, which determine the position of a point, be 7, 5 and 8 inches respectively; taking from a scale, $OA=7$, $OB=5$ and $OC=8$ (fig. 2), and drawing AD, DE parallel and equal to OB, OC respectively, E is the position of the given point.

Completing the construction in the figure, we have the representation of a rectangular parallelopiped such that each edge may be measured to a given scale.

7. We should remark here, that in the figure, the lines are to a scale of an eighth of their true length, but that, to such a scale, the point E is not the orthographic projection of the given point, with respect to the point O. If it were required to give its projection to such a scale, we must take

$$Oa=7\cdot\frac{1}{3}\sqrt{6}=5{\cdot}7157,\ Ob=5\cdot\frac{1}{3}\sqrt{6}=4{\cdot}0825,\ Oc=8\cdot\frac{1}{3}\sqrt{6}=6{\cdot}5320;$$

then constructing as before, *e* will be the true orthographic projection of the given point with respect to the point O, to a scale of an eighth. For the purposes of construction it is however better to consider the distances on the axes OA, OB, OC as taken of their true length on a scale.

8. The axes OX, OY, OZ are called the *Isometric Axes*, because lines in them are all measured to the same scale; and for the same reason all lines parallel to any one of these axes are called *Isometric Lines*.

9. One of the rectangular planes XOY being taken as the horizontal plane, the other two XOZ, YOZ are vertical planes: these planes may be designated as the horizontal plane XY, the vertical plane ZX, the vertical plane ZY. These and all planes parallel to any one of them are called *Isometric Planes*.

In the representation of objects, such as buildings, portions of a fortification, &c., the plan is represented in the horizontal plane XY, and elevations or sections in the vertical plane ZX, ZY, or planes parallel to these.

10. The point in the object which is assumed as the origin of the axes is called the *Regulating Point*.

11. If any point in the same isometric plane with the point required to be found be already represented in the picture, that point may be assumed as a new regulating point, and the point required be found by taking two distances, from the scale of the picture; and if the new assumed regulating point be in the same isometric line with the required point, it is found by taking only one distance.

12. It is evident that by means of the isometric projection or perspective, any objects, such as buildings or frame-work, the lines in which are in the isometric directions, may be easily and accurately represented to any scale required; and that by means of such a picture those objects may be readily and correctly constructed, without further explanation.

13. To represent lines which are not in the isometric directions, it is only necessary to determine the isometric projections of their extremities, and to join them, the positions of these extremities being supposed to be given with reference to the isometric axes, or some assumed regulating point.

14. Curved lines may be represented by determining the co-ordinates of a sufficient number of points, and taking these co-ordinates to a scale in the direction of the isometric axes.

15. In representing astronomical and other instruments, machinery in which there are wheels working into each other, and, frequently, buildings, there are circles to be represented which are very commonly in the isometric planes, and it is a matter of much convenience that the picture of all circles in these planes is an

ellipse* of the same form, in whichever of the three isometric planes the circle may be.

16. If circles be inscribed on the faces of the cube to which we at first referred, it is evident that, in each, the projection of that diameter which is in the direction of the diagonal parallel to the plane of projection is the greatest; and the projection of the diameter at right angles to this, and which is in the direction of the inclined diagonal, is the least: the former is therefore the major axis of the ellipse in which the circle is projected, and the latter is its minor axis. The projections of these diameters being to each other as the projections of the corresponding diagonals, we have the major axis of the ellipse to its minor axis as $\sqrt{2}$ to $\frac{1}{3}\sqrt{6}$, or as 1 to $\frac{1}{3}\sqrt{3}$. An edge of the cube being 1, the major axis of the ellipse, which is equal to the diameter of the inscribed circle, is also 1, and consequently the minor axis is $\frac{1}{3}\sqrt{3}$.

17. The diameters of the ellipse which are the projections of diameters of the circle parallel to the edges of the cube, and are therefore parallel to the isometric axes, are called *Isometric Diameters*. Being equal to the projection of the edge, they are equal to $\frac{1}{3}\sqrt{6}$.

18. It follows from this, that—

Taking the diameter of the circle to be represented as 1, the minor axis, the isometric diameter and the major axis of the ellipse are $\frac{1}{3}\sqrt{3}$, $\frac{1}{3}\sqrt{6}$ and 1 respectively.

Taking the isometric diameter as 1, the minor axis, the isometric diameter and the major axis are $\frac{1}{2}\sqrt{2}$, 1 and $\frac{1}{2}\sqrt{6}$ respectively.

Taking the minor axis as 1, the isometric diameter and major axis are respectively $\sqrt{2}$ and $\sqrt{3}$†.

* It may be necessary here to state, that if a cone or a cylinder be cut obliquely through both sides by a plane, the section is called an *Ellipse*. The two straight lines by which an ellipse is divided into parts which are perfectly symmetrical on each side of them, are called its *Axes*, the longer being called the *Major Axis*, and the shorter the *Minor Axis*. An ellipse is readily thus described: two pins are fixed upright on the paper; a thread having its two ends tied together is then passed over the pins; this thread is stretched tight by a pencil point, and a curve is traced by the pencil on the paper round the pins, keeping the thread constantly stretched: the curve thus traced is an ellipse. The species of ellipse so described depends upon the ratio which the length of the thread bears to the distance between the pins. The ordinate in an ellipse has a constant ratio to the corresponding ordinate in a circle; and as this is the case with the ordinate of a circle and the orthographic projection of that ordinate, it follows that the orthographic projection of a circle is an ellipse.

† To describe, by the method pointed out, an ellipse of the above form, and

19. In fig. 3 are represented circles as projected in ellipses on the isometric planes with their major and minor axes, and isometric diameters. If such circles represent wheels in machinery, or circles having axles in an instrument, these axles being perpendicular to the isometric planes, must be in the direction of the minor axes of the ellipses.

20. If the circle to be represented be graduated, that is, have its circumference divided into any number of equal parts, these graduations may be correctly represented on the ellipse. Let AG (fig. 4) be the axis major of the ellipse A*e*G representing the circle upon the isometric plane; then the circle AEG described upon AG as a diameter, will be that circle in its true dimensions (17). Let B, C, D be the points of graduation in the circle: then since the diameter AG is parallel to the plane of projection, the projections of B, C, D will be in planes perpendicular to AG, and passing through B, C, D; and consequently will be *b*, *c*, *d*, the intersections of the perpendiculars to AG drawn from B, C, D, with the ellipse A*e*G representing the circle on the plane of projection.

21. By means of a series of these ellipses, we obtain scales on which distances in other directions than the isometric may be measured. For this purpose it is only necessary to take the intervals between the ellipses, in the isometric directions from the centre, equal to the units on the isometric scale, as in fig. 5; the intersections of these ellipses with any line drawn through the centre will be at intervals which are the units on that line, and will therefore form the scale for that line and for all lines which have the same position with reference to the isometric lines. Thus distances measured on the longer diagonals AF, FD, DA (fig. 1) of these squares, or on lines parallel to them, are to be measured by the divisions on the major axis; and those on the shorter diagonals OG, OE, OB, or their parallels, by divisions on the minor axis: and so in other directions.

22. For the purpose of drawing lines parallel to the isometric axes, and at required distances from a regulating point, a long flat rectangular ruler, having its edges on each side graduated from their middle points and to different scales, with an *equilateral* triangle, having an index mark in the middle of each side, may be most conveniently employed.

23. In applying the principles of isometric perspective to the representation of objects, it is necessary to bear in mind the assump-

having a given isometric diameter, the distance between the pins must be taken equal to the given diameter, and the length of the string must be to the distance between the pins as $1+\frac{1}{2}\sqrt{6}$ is to 1, or as 2·225 to 1. In practice it may be best to double the string, and tie it so that the length to the knot shall be to half the distance between the pins, as 2·225 to 1.

tion which may be made regarding the scale to which the drawing is adapted, in conformity with what has already been stated (7).

When the drawing is to be considered merely as representing the object to a scale, for instance to one of $\frac{1}{100}$ of the real dimensions, then dimensions represented in the directions of the isometric axes are to be taken on a scale of $\frac{1}{100}$, and applied at once to those axes. In this case the isometric diameters of any ellipses will be the diameters of the circles they represent taken on the scale of $\frac{1}{100}$.

When, however, the drawing is to be considered as the isometric *projection* of a model of the object on a reduced scale of $\frac{1}{100}$, then dimensions in the directions of the isometric axes are to be taken on a scale of $\frac{1}{100}\cdot\frac{1}{3}\sqrt{6}$. Such a scale may be readily constructed by drawing a horizontal line, and another inclined to it at an angle of 35° 16′; constructing the given scale on the inclined line, and from the points of division letting fall perpendiculars on the horizontal line: the horizontal scale will be the one required.

The manner of applying the principles of isometric perspective to the representation of objects will be sufficiently illustrated by fig. 6 (Pl. XII.), which represents a Powder Magazine.

GEOMETRY OF PLANES AND SOLIDS.

(CONTINUED.)

GEOMETRY OF SOLIDS.

PROP. XLIX. THEOR.

Two solids which are contained by the same number of equal and similar plane figures, similarly situated, and which have none of their solid angles contained by more than three plane angles, are equal and similar to one another.

Let the solids AI, NW (fig. 54), which have none of their solid angles contained by more than three plane angles, be contained by the same number of equal and similar plane figures similarly situated, viz. the plane figure ABCDE equal and similar to NOPQR; also ABGFM to NOTSZ; BH to OV; CI to PW; DK to QX; EKLMA to RXYZN; and LMF to YZS: the solid AI shall be equal and similar to the solid NW.

Since the plane figures which contain the two solids are equal, similar and similarly situated, the plane angles which contain the solid angles of the one are respectively equal to the plane angles which contain the corresponding solid angles in the other: the angles ABC, ABG, GBC severally equal to the angles NOP, NOT, TOP; and the same of the angles which contain the other solid angles. And since the plane angles which contain the solid angle B are equal to the plane angles which contain the solid angle O, the dihedral angles contained by the faces about the angle B, will be equal to the dihedral angles contained by the equal faces about the angle O (Prop.47). If, therefore, the solid AI be applied to the solid NW, so that the plane figure ABCDE may coincide with NOPQR to which it is equal and similar, the point B coinciding with O, and AB, BC coinciding with NO, OP to which they are equal, the plane figures ABGFM and GBCH will coincide with NOTSZ and TOPV to which they are respectively equal and similar, the dihedral angles formed by the planes ABGFM and GBCH with the plane ABCDE, and with each other, being respectively equal to the dihedral angles formed by the planes NOTSZ and TOPV, with the plane NOPQR, and with each other. In the same manner it may be shown that

the figure HCDI will coincide with VPQW; IDEK with WQRX; KEAML with XRNZY; FGHIKL with STVWXY; and LMF with YZS: and the solid AI will coincide with the solid NW, and be equal to it: which was to be proved.

PROP. L. THEOR.

Two prisms which have a solid angle in the one, contained by three plane figures, equal, similar and similarly situated to the three plane figures about a solid angle of the other, are equal and similar.

Let the two prisms AI, LT (fig. 55), have the solid angle at B, in the one, contained by the three plane figures ABCDE, ABGF, GBCH equal, similar and similarly situated to the three plane figures LMNOP, LMRQ, RMNS about the solid angle at M, in the other; the prism AI shall be equal and similar to the prism LT.

Because the plane figures about the solid angle at B are equal, similar and similarly situated to the figures about the solid angle at M, the three plane angles of the one solid angle are equal to the three plane angles of the other, each to each, and therefore the dihedral angles contained by the faces of the one are equal to the dihedral angles contained by the faces equal to them of the other (Prop. 47), the angle EABG equal to the angle PLMR, the angle ABCH equal to the angle LMNS and the angle ABGH equal to the angle LMRS. And because the plane angles BAE, BAF, of the solid angle at A, are equal to the plane angles MLP, MLQ, of the solid angles at L, each to each, and the dihedral angles EABG, PLMR, formed by the planes, are also equal, the third plane angles FAE, QLP of the solid angles at A and L are equal (Prop. 48). The parallelograms AK, LV have therefore the two sides FA, AE equal to the two sides QL, LP, each to each, and the angle FAE equal to the angle QLP; consequently the parallelogram AK is equal and similar to the parallelogram LV.

In the same manner it may be shown that the parallelogram EI is equal and similar to the parallelogram PT, and the parallelogram DH equal and similar to the parallelogram OS.

And because the figure FGHIK is equal and similar to ABCDE (Def. 15), and QRSTV equal and similar to LMNOP, and that ABCDE and LMNOP are equal and similar, FGHIK is equal and similar to QRSTV (VI. 21).

Since, therefore, the prisms AI and LT are contained by the same number of equal and similar plane figures similarly situated, they are equal and similar to one another (Prop. 49): which was to be proved.

Cor. It follows from this that if two prisms AI, LT upon equal and similar bases ABCDE, LMNOP, have a bounding parallelogram AG standing upon the side AB of the base of the one, equal and similar, and similarly situated to the bounding parallelogram LR

standing upon the equal and homologous side LM of the base of the other, and have the dihedral angles EABG, PLMR, contained by these parallelograms and the bases also equal, they are equal and similar to one another; for in this case the angle FAE will be equal to the angle QLP (Prop. 48), and therefore the parallelogram AK equal, similar, and similarly situated to the parallelogram LV.

Prop. LI. Theor.

If a solid be contained by six planes, two and two of which are parallel, the opposite planes are similar and equal parallelograms.

Let the solid ABFE (fig. 56) be contained by the parallel planes AC, GF; BG, CE; FB, AE: its opposite planes shall be similar and equal parallelograms.

Because the two parallel planes BG, CE, are cut by the plane AC, their common sections AB, CD, are parallel (Prop. 27): again, because the two parallel planes BF, AE are cut by the plane AC, their common sections AD, BC are parallel (Prop. 27): and AB is parallel to CD; therefore AC is a parallelogram. In like manner, it may be proved that each of the figures CE, FG, GB, BF, AE, is a parallelogram.

Join AH, DF; and because AB is parallel to DC, and BH to CF, the two straight lines AB, BH, which meet one another, are parallel to DC and CF, which meet one another: wherefore they contain equal angles (Prop. 20); therefore the angle ABH is equal to the angle DCF. And because AB, BH, are equal to DC, CF, each to each, and the angle ABH equal to the angle DCF; therefore the base AH is equal to the base DF (I. 4), and the triangle ABH to the triangle DCF: but the parallelogram BG is double of the triangle ABH (I. 34), and the parallelogram CE double of the triangle DCF; therefore the parallelogram BG is equal and similar to the parallelogram CE. In the same manner it may be proved, that the parallelogram AC is equal and similar to the parallelogram GF, and the parallelogram AE to BF. Therefore, if a solid, &c.: which was to be proved.

Prop. LII. Theor.

The opposite dihedral angles of a parallelepiped are equal; and its opposite trihedral angles are symmetrical.

Let ABFE (fig. 56) be a parallelepiped; its opposite dihedral angles are equal, viz. the dihedral angles GABC and CFEG; GADC and CFHG; BAGE and EFCB; ABHF and FEDA; ABCF and FEGA; AGHF and FCDA; and its opposite trihedral angles are symmetrical, viz. the solid angles at A and F; at C and G; at D and H; and at B and E.

Because the parallelograms HA and DF are equal, and similar (Prop. 51), the angle BAG is equal to the angle CFE; and for the like reason the angle BAD is equal to the angle EFH; and the angle GAD to the angle CFH. The trihedral angles at A and F have therefore the three faces of the one equal to the three faces of the other, each to each, and consequently the dihedral angles contained by the faces of the one are equal to the dihedral angles contained by the faces equal to them of the other (Prop. 47): therefore the dihedral angles GABC and CFEG, GADC and CFHG, BAGE and EFCB are respectively equal.

In like manner it may be shown that the other opposite dihedral angles of the parallelepiped ABFE are equal.

The opposite trihedral angles at A and F are symmetrical, because the three faces of the one are equal to the three of the other, each to each, but the equal faces do not follow the same order, in the same direction, about the two solid angles (Prop. 47. Scholium).

In like manner it may be shown that the other opposite trihedral angles of the parallelepiped ABFE are symmetrical.

Wherefore the opposite dihedral angles of a parallelepiped are equal, &c.: which was to be proved.

Prop. LIII. Theor.

If a parallelepiped be cut by a plane parallel to two of its opposite planes, it will be divided into two parallelepipeds, which will be to one another as their bases.

Let the parallelepiped ABCD (fig. 57) be cut by the plane EK which is parallel to the opposite planes AL, HD; it divides the whole ABCD into the two parallelepipeds ABFK, EGCD; and as the base AF is to the base HF, so is the solid ABFK to the solid EGCD.

Let the edges AH, MC, BI, LD be produced both ways; in AH produced take any number of lines AN, NO, OP, each equal to EH, and any number HQ, QR, each equal to EH; and through A, N, O, P, Q, R, let the planes NS, OT, PU, QV, RW be drawn parallel to the opposite planes AL, HD, and constituting with the other opposite planes produced of the parallelepiped ABCD, the parallelepipeds NL, OS, PT, HV, QW. Then the bases of the parallelepipeds PT, OS, NL, AK are all equal (I. 36) and similar, and their respective opposite planes are all equal and similar (Prop. 51); also the dihedral angles contained by the corresponding planes are equal (Prop. 40). If therefore the parallelepiped AK were applied to any one of the parallelepipeds PT, OS, NL, so that their bases should coincide, the two solids would coincide and be equal to each other; consequently the parallelepipeds PT, OS, NL, AK are equal to one another, and therefore whatever multiple the base PF is of the base

AF, the same multiple is the solid PK of the solid AK. In like manner it may be shown that the parallelepipeds QW, HV, ED are equal to one another, and therefore whatever multiple the base RF is of the base HF, the same multiple is the solid RK of the solid HK. And if the solid PK be applied to the solid RK so that their bases coincide, if the base PF be equal to the base RF, the solid PK will be equal to the solid RK; and if the base PF be greater than the base RF, the solid PK will be greater than the solid RK; and if less, less. Since then there are four magnitudes, the two bases AF, HF and the two solids AK, HK; and that of the base AF and the solid AK, the base PF and the solid PK are any equimultiples whatever; and of the base HF and the solid HK, the base RF and the solid RK are any equimultiples whatever; and it has been shown that if the base PF be equal to the base RF, the solid PK will be equal to the solid RK; if greater, greater; and if less, less; therefore (V. Def. 5) as the base AF is to the base HF, so is the solid AK to the solid HK. Wherefore if a parallelepiped, &c.: which was to be proved.

Cor. Because the parallelogram AF is to the parallelogram HF as MF to FC, the solid AK is to the solid HK as MF to FC.

Prop. LIV. Theor.

If a parallelepiped be cut by a plane passing through the diagonals of two of its opposite planes, it will be cut into two equal prisms.

Let AD (fig. 58) be a parallelepiped, and AF, HD the diagonals of the opposite parallelograms BE, GC, namely, those which are drawn between the equal angles in each. And because AH and FD are each of them parallel to EC, AH and FD are parallel (Prop. 19); wherefore the diagonals AF, HD, are in the plane in which the parallels are, and are themselves parallels (Prop. 27): the plane AFDH will cut the solid AD into two equal prisms.

Produce the parallels AH, EC, FD indefinitely towards H, C, D; in EC produced take $CG_{,,}$ equal to EC or GB; in EC at the point $G_{,,}$ draw $G_{,}H_{,}$ in the plane $FG_{,,}$ making the angle $EG_{,}H_{,}$ equal to the angle BGH, and meeting FD produced in $H_{,}$; and, in the plane $AG_{,,}$ draw $G_{,}D_{,}$ making the angle $EG_{,}D_{,}$ equal to the angle BGD, and meeting AH produced in $D_{,}$: complete the parallelograms $CH_{,,}$ $CD_{,,}$ by drawing through C, $CA_{,,}$ parallel to $G_{,}H_{,,}$ and $CF_{,}$ parallel to $G_{,}D_{,}$; and the parallelogram $F_{,}H_{,}$ by joining $H_{,}D_{,}$ and $A_{,}F_{,}$ (I. 33). The plane $A_{,}CF_{,}$ and $H_{,}G_{,}D_{,}$ are therefore parallel (Prop. 26).

Because the angle $CG_{,}H_{,}$ is equal to the angle BGH the parallelogram $A_{,}G_{,}$ is equiangular to the parallelogram AG (I. 29. 34), and therefore to the parallelogram ED (Prop. 51). And because the angle $A_{,}DC$ is equal to ECD (I. 29), and the angle $DA_{,}C$ to $A_{,}CG_{,}$; and that ECD has been shown to be equal to $A_{,}CG_{,}$;

$A_{,}DC$ is equal to $DA_{,}C$; and consequently $A_{,}C$ is equal to DC (I. 6). In the same manner it may be shown that $CF_{,}$ is equal to CH, and $CG_{,}$ was made equal to EC; therefore the parallelograms $CH_{,}$ and CF have the sides about their equal angles equal, each to each; consequently the parallelogram $CH_{,}$ is equal and similar to the parallelogram CF, and therefore to the parallelogram BH. For the same reason the parallelogram $CD_{,}$ is equal and similar to the parallelogram CA, and therefore to the parallelogram BD.

And because the trihedral angles at G and $G_{,}$ have the plane angles BGH, BGD equal to the plane angles $CG_{,}H_{,}$, $CG_{,}D_{,}$, each to each, and the dihedral angles FBGH, $F_{,}CG_{,}H_{,}$ also equal (Prop. 52), their third plane angles DGH, $D_{,}G_{,}H_{,}$ are equal, and the dihedral angle AHGD is equal to the dihedral angle $A_{,}H_{,}G_{,}D_{,}$, and FDGH to $F_{,}D_{,}G_{,}H_{,}$ (Prop. 48); also the solid angles at G and $G_{,}$ are equal, the equal plane angles about them being similarly situated (Prop. 47. sch.). The triangles DGH, $D_{,}G_{,}H_{,}$ have therefore the two sides DG, GH equal to the two sides $D_{,}G_{,}$, $G_{,}H_{,}$, each to each, and the angles DGH, $D_{,}G_{,}H_{,}$ also equal, consequently they are equal and similar (I. 4).

The two prisms ABFHGD, $A_{,}CF_{,}H_{,}G_{,}D_{,}$ have therefore the solid angle at G, in the one, contained by three plane figures equal, similar and similarly situated to the three plane figures about the solid angle at $G_{,}$, in the other, consequently the prism ABFHGD is equal and similar to the prism $A_{,}CF_{,}H_{,}G_{,}D_{,}$ (Prop. 50).

And because AH is equal to $F_{,}D_{,}$, $AF_{,}$ is equal to $HD_{,}$; therefore the quadrilaterals $AF_{,}CE$, $HD_{,}G_{,}C$, which are equiangular, have their sides about their equal angles also equal, each to each; consequently they are similar (VI. Def. 1) and equal (VI. 20). In the same manner, FD being equal to $A_{,}H_{,}$, $FA_{,}$ is equal to $DH_{,}$; the equiangular quadrilaterals $FA_{,}CE$, $DH_{,}G_{,}C$ have their sides about their equal angles also equal, each to each, and are similar and equal; and the equiangular quadrilaterals $AFA_{,}F_{,}$, $HDH_{,}D_{,}$ are similar and equal. The two solids $FEAA_{,}CF_{,}$, $DCHH_{,}G_{,}D_{,}$ are therefore contained by the same number of equal and similar plane figures, similarly situated; and are consequently equal (Prop. 49). From each of these equal solids take away the solid $DCHA_{,}CF_{,}$, which is common to both, and there remains of the prism FEADCH equal to the prism $A_{,}CF_{,}H_{,}G_{,}D_{,}$. But the prism ABFHGD has been shown to be equal to the prism $A_{,}CF_{,}H_{,}G_{,}D_{,}$; therefore the prism ABFHGD is equal to the prism FEADCH; and the parallelepiped AD is divided into two equal prisms by the plane AFDH: which was to be proved.

PROP. LV. THEOR.

Parallelepipeds which are upon the same base, and have their planes opposite to the base in the same plane, are equal to one another.

Let the two parallelepipeds AK, BF (figs. 59, 60, 61), be upon the same base AB, and have their planes GK, FH, opposite to the base AB, in the same plane: the solid AK is equal to the solid BF.

First, let the parallelograms GK, FH (figs. 59, 60) opposite to the base AB have two of their sides DH, EK in the same straight line, and therefore be between the same parallels DK, FN.

If the parallelograms GK, FH have a common side GH (fig. 59); then, because the parallelepiped AK is cut by the plane BHGL passing through the diagonals BH, GL of its opposite planes BCHK, LAGN, AK is cut into two equal parts by the plane BHGL (Prop. 54); therefore the parallelepiped AK is double of the prism which is contained between the triangles CBH, ALG: and because the parallelepiped BF is cut by the plane CHGA passing through the diagonals CH, AG of its opposite planes BCDH, LAFG, the parallelepiped BF is double of the same prism contained between the triangles CBH, ALG. Consequently the parallelepiped AK is equal to the parallelepiped BF.

If the parallelograms GK, FH (fig. 60) have not a common side; then because CK, BD are parallelograms, CB is equal to each of the opposite sides EK, DH (I. 34); and therefore EK is equal to DH: add or take away HE, then HK is equal to DE: also BK is equal to CE (I. 34), and the angle HKB is equal to the angle DEC (I. 29): therefore the triangles HBK, DCE are equal (I. 4) and likewise similar (VI. Def. 1). For the same reason the triangles MLN, FAG are equal and similar. And because HM and KN are each parallel to BL, HM and KN are parallel (Prop. 19); and HN is a parallelogram. For the same reason DF is parallel to GE, and DG is a parallelogram. The parallelograms HN and DG are equal, because HK is equal to DE (I. 36); and they are similar, being equiangular. Also the parallelograms LK, AE are equal and similar (Prop. 51); and likewise the parallelograms BM, CF. The prism which is contained by the two triangles HBK, MLN and the three parallelograms HN, LK, BM is therefore equal to the prism which is contained by the two triangles DCE, FAG and the three parallelograms DG, AE, CF (Prop. 49). If therefore from the prism whose ends are the trapezoids CBKD, ALNF, and sides are the parallelograms AB, BN, ND, DA, there be taken the triangular prism CEDAGF, and from this same solid there be taken the triangular prism BKHLNM, the remaining parallelepiped AK will be equal to the remaining parallelepiped BF.

Next let the parallelograms GK, FH (fig. 61), opposite to the

base AB of the two parallelepipeds, not have two of their sides in the same straight line.

Produce FD, MH and KE, NG, and let them meet each other in the points R, Q, O, P; join AO, CR, BQ, LP.

Because the planes LBHM and ACDF are parallel; and the plane LBHM is that in which are the parallels LB, MHPQ, and in which also is the figure BLPQ; and the plane ACDF is that in which are the parallels AC, FDOR, and in which also is the figure CAOR; the figures BLPQ, CAOR are in parallel planes. In like manner, because the planes ALNG and CBKE are parallel; and the plane ALNG is that in which are the parallels AL, OPGN, and in which also is the figure ALPO; and the plane CBKE is that in which are the parallels CB, RQEK, and in which also is the figure CBQR; the figures ALPO, CBQR are in parallel planes. But the planes ACBL, ORQP are also parallel; therefore the solid CP is a parallelepiped (Def. 32). Now, by the first case, the parallelepiped AK is equal to the parallelepiped CP, since they are upon the same base AB, and the parallelograms GK and OQ opposite to the base have two of their sides KE, QR in the same straight line; and for the like reason, the parallelepiped BF is equal to the parallelepiped CP: consequently the parallelepiped AK is equal to the parallelepiped BF.

Wherefore parallelepipeds, which are upon the same base, &c.: which was to be proved.

Cor. 1. Parallelepipeds upon the same base and of the same altitude are equal, for their altitude being the same, their sides opposite to the base will be in the same plane.

Cor. 2. Any parallelepiped is equal to a "Right" parallelepiped, (that is, a parallelepiped having all the planes adjacent to the base at right-angles to it,) constituted upon the same base, between the same parallel planes, or of the same altitude.

Prop. LVI. Theor.

Parallelepipeds which are upon equal bases and of the same altitude are equal to one another.

Let the parallelepipeds AE, CF (figs. 62, 63, 64) be upon equal bases AB, CD, and of the same altitude; that is, have the perpendiculars between the planes AB, GE and CD, NF equal to one another: the solid AE is equal to the solid CF.

Case 1.—Let AE, CF (figs. 62, 63) be right parallelepipeds, that is, have the planes AK, HE, BM, LG and CM, LF, DP, ON, adjacent to their bases AB and CD, at right angles to them; and let the bases be placed in the same plane, and so that the sides CL, LB be in a straight line. The planes CM, DM being at right

angles to the base CD, their intersection LM is at right angles to the plane CD (Prop. 34); and for the same reason, the intersection of the planes AM, BM is at right angles to the plane AB, that is to the plane CD; consequently, LM is also the intersection of the planes AM, BM (Prop. 8).

First, let the angle ALB be equal to the angle CLD (fig. 62); then AL, LD are in a straight line (I. 14). Produce OD, HB, and let them meet in Q; and complete the parallelepiped LR, of which the base is LQ.

Because the parallelogram AB is equal to the parallelogram CD, as the base AB is to the base LQ, so is the base CD to the base LQ (V. 7). And because the parallelepiped AR is cut by the plane LMEB, which is parallel to the opposite planes AK, DR; as the base AB is to the base LQ, so is the solid AE to the solid LR: for the same reason, because the parallelepiped CR is cut by the plane LMFD which is parallel to the opposite planes CP, BR; as the base CD is to the base LQ, so is the solid CF to the solid LR: but it has been shown that as the base AB is to the base LQ, so is the base CD to the base LQ; therefore as the solid AE is to the solid LR, so is the solid CF to the solid LR (V. 11): consequently the solid AE is equal to the solid CF (V. 9).

Next let the angles ALB, CLD of the equal bases AB, CD, be unequal (fig. 63). Produce HA, DL until they meet in S; from B draw BT parallel to DS; and let TB, OD produced meet in Q; and complete the parallelepipeds LX, LR: then the parallelepiped SE, of which the base is the parallelogram LE, and SX the plane opposite, is equal to the parallelepiped AE, of which the base is LE, and AK the plane opposite; for they are upon the same base, and the parallelograms have two of their sides opposite to the base in the same straight line VK (Prop. 55). And because the parallelogram SB is equal to the parallelogram AB (I. 35); and that the base AB is, by hypothesis, equal to the base CD; the base SB is equal to the base CD: but the angle SLB is equal to the angle CLD (I. 15); therefore by the former case the solid SE is equal to the solid CF: and it has been shown that the solid SE is equal to the solid AE; therefore the solid AE is equal to the solid CF.

Case 2.—Let the planes AK, HE, BR, QG and CM, LF, DP, ON (fig. 64) of the parallelepipeds AE and CF, adjacent to their bases AB and CD, be not at right angles to them: in this case likewise the parallelepiped AE is equal to the parallelepiped CF.

Because parallelepipeds which are upon the same base, and have their planes opposite to the base in the same plane, are equal (Prop. 55), if two parallelepipeds be constituted on the bases AB and CD by planes at right angles to them, and have their planes opposite to the bases AB and CD in the same planes as GE and NF respectively, they will be equal to the solids AE and CF re-

spectively, and, by the first case, they will be equal to one another; therefore the parallelepipeds AE and CF are also equal to one another.

Wherefore parallelepipeds which are upon equal bases, &c.: which was to be proved.

Cor. Any parallelepiped is equal to a rectangular parallelepiped, (that is, a parallelepiped every two adjacent planes of which are perpendicular to each other,) upon an equal base and of the same altitude.

Prop. LVII. Theor.

Parallelepipeds which have the same altitude are to one another as their bases.

Let AB, CD (fig. 65) be parallelepipeds which have the same altitude: they are to one another as their bases; that is,

The solid AB is to the solid CD, as the base AE to the base CF.

To the straight line FG apply the parallelogram FH equal to AE, and having the angle FGH equal to the angle LCG (I. 45. *Cor.*), so that CG, GH is a straight line; and complete the parallelepiped GK between the planes CFH, PDM, so that GK is of the same altitude as CD, and therefore as AB. Then the solid GK is equal to the solid AB, because they are upon equal bases and have the same altitude (Prop. 56). And because the parallelepiped CK is cut by the plane DG, parallel to its opposite planes, the solid GK is to the solid CD as the base FH to the base CF: but the solid GK is equal to the solid AB, and the base FH to the base AE; therefore the solid AB is to the solid CD as the base AE to the base CF. Wherefore parallelepipeds which have the same altitude, &c.: which was to be proved.

Prop. LVIII. Theor.

Parallelepipeds are to one another in the ratio that is compounded of the ratios of their bases, and of their altitudes.

Let AF, GO (fig. 66) be two parallelepipeds, of which the bases are AC and GK, and the altitudes, the perpendiculars let fall on the planes of these bases from any point in the opposite planes EF and MO: the solid AF is to the solid GO in the ratio compounded of the ratios of the base AC to the base GK, and of the perpendicular on AC to the perpendicular on GK.

Case 1.—Let the solids AF, GO be right parallelepipeds, so that AE and GM, being at right angles to the bases AC and GK, are the altitudes of the solids. In GM take GQ equal to AE, and through Q let a plane pass parallel to GK, meeting the planes GN, HO, KP, LM in QR, RS, ST, TQ. GS is therefore a parallelepiped, and it has the same altitude as AF, viz. GQ or AE.

To GH apply the parallelogram HX equal to AC, so that GLX

is a straight line (I. 45. *Cor.*), and complete the parallelepiped GV, which is equal to the parallelepiped AF, because they are upon equal bases HX, AC, and of equal altitudes GQ, AE (Prop. 56). Take GY a fourth proportional to GQ or AE, GM, GL (VI. 12), that is GQ or AE to GM as GL to GY.

Because GX is to GL as the parallelogram HX to the parallelogram GK (VI. 1), and that HX is equal to the base AC, the ratio GX to GL is the ratio of the base AC to the base GK; also GL to GY is the ratio of the altitude AE to the altitude GM. Now the ratio of GX to GY is the ratio compounded of the ratios of GX to GL and GL to GY (V. Def. 11. A.); consequently the ratio of GX to GY is the ratio compounded of the ratios of the base AC to the base GK, and of the altitude AE to the altitude GM. Also the solid GV is to the solid GS, as the base HX to the base GK (Prop. 57), that is as GX to GL; and since the solid GV is equal to the solid AF, the solid AF is to the solid GS as GX to GL. But the solid GS is to the solid GO as GQ to GM (Prop. 57), that is as GL to GY. Since then

the solid AF : the solid GS :: GX : GL
and the solid GS : the solid GO :: GL : GY
ex æquali, the solid AF : the solid GO :: GX : GY (V. 22).

But it has been shown that GX has to GY the ratio compounded of the ratios of the base AC to the base GK, and of the altitude AE to the altitude GM; therefore the solid AF has to the solid GO, the ratio compounded of the ratios of the base AC to the base GK, and of the altitude AE to the altitude GM.

Case 2.—When the solids are not right parallelepipeds.

Let the parallelograms AC and GK be the bases, and AE, GM the altitudes, of two oblique parallelepipeds W, Z. Then, if the right parallelepipeds AF, GO be constituted on the bases AC, GK, with the altitudes AE and GM, they will be equal to the parallelepipeds W, Z. Now by the first case the solid AF has to the solid GO the ratio compounded of the ratios of the base AC to the base GK, and of the altitude AE to the altitude GM; therefore also the oblique parallelepiped W has to the oblique parallelepiped Z the ratio compounded of the same ratios.

Wherefore parallelepipeds are to another, &c.: which was to be proved.

Scholium. Since arithmetically the ratio compounded of two ratios m to n and p to q is the ratio $m \times p$ to $n \times q$ (Alg. 293), if A and a be numbers proportional to the altitudes of two parallelepipeds P and p, and B and b numbers proportional to the areas of their bases, then by this proposition, $P : p :: A \times B : a \times b$.

Also since the parallelograms which are the bases of the solids are equal to rectangles having the same base and altitude as the parallelograms, and that these rectangles are to each other in the

ratio compounded of the ratios of their sides (VI. 23), if C and c be the bases, and D and d the altitudes of the parallelograms B and b,
then $B : b :: C \times D : c \times d$
and consequently $P : p :: A \times C \times D : a \times c \times d$.

Prop. LIX. Theor.

Similar parallelepipeds are to one another in the triplicate ratio of their homologous edges.

Let AG, KQ (fig. 67) be two similar parallelepipeds, of which AB and KL are two homologous sides: the ratio of the solid AG to the solid KQ is triplicate of the ratio of AB to KL.

Because the solids are similar, the parallelograms AF, KP are similar (Def. 13), as also the parallelograms AH, KR; therefore the ratios of AB to KL, AE to KO and AD to KN are all equal (VI. Def. 1). But the ratio of the solid AG to the solid KQ is compounded of the ratios of AC to KM and AE to KO; and the ratio of AC to KM is compounded of the ratios of AB to KL and AD to KN, because AC and KM are similar parallelograms (VI. 23); consequently the ratio of the solid AG to the solid KQ is compounded of the three ratios of AB to KL, AD to KN, and AE to KO. But these three ratios are equal; therefore the ratio that is compounded of them is triplicate of any one of them, and consequently the ratio of the parallelepiped AG to the parallelepiped KQ is the triplicate of the ratio of AB to KL. Wherefore similar parallelepipeds, &c.: which was to be proved.

Cor. As cubes are similar parallelepipeds, the cube on AB is to the cube on KL in the triplicate ratio of AB to KL, that is, in the same ratio as the solid AG to the solid KQ. Similar parallelepipeds are therefore to one another as the cubes on their homologous sides.

Prop. LX. Theor.

If a prism be cut by parallel planes the sections are equal and similar plane figures.

Let the prism ABCDEFGHIK (fig. 68) be cut by the parallel planes LO and QT, the sections LMNOP and QRSTV are equal and similar plane figures.

Because the parallel planes LO and QT are cut by the plane AG, the sections LM and QR are parallel (Prop. 26); and LQ and MR are parallel (Def. 15): therefore LMRQ is a parallelogram, and LM is equal to QR. In like manner it may be shown that MN, NO, OP, PL are respectively parallel and equal to RS, ST, TV, VQ. And because LM and MN are parallel to QR and RS, the angle LMN is equal to the angle QRS (Prop. 20). In like manner it may be shown that the angles MNO, NOP, OPL, PLM are severally equal

to the angles RST, STV, TVQ, VQR; consequently the figures LMNOP and QRSTV are equiangular; and they have the sides about their equal angles equal each to each: they are therefore equal and similar. Wherefore if a prism be cut by parallel planes, &c.: which was to be proved.

Prop. LXI. Theor.

If a prism, having a triangle for its base, and a parallelepiped, be upon equal bases and have equal altitudes, they shall be equal to one another.

Let the prism ACE (fig. 69) having the triangle ABC for its base, and the parallelepiped GH be upon equal bases ABC and GK, and have equal altitudes, namely the perpendiculars let fall upon the planes ABC and GK from any point in the opposite planes DEF and IH; the prism ACE is equal to the parallelepiped GH.

Bisect AC in L, draw BN parallel to AC, and LM, CN parallel to AB; complete the parallelepiped BF; and through LM let the plane LP pass parallel to AH, dividing the parallelepiped into two equal parallelepipeds BO, MF (Prop. 53. Cor.).

And because the prism ACE is half the parallelepiped BF (Prop. 54), and that the parallelepiped BO is also half the parallelepiped BF, the prism ACE is equal to the solid BO. But the parallelogram BL is equal to the triangle ABC (I. 42); and the parallelogram GK is by hypothesis equal to the triangle ABC; therefore the base GK is equal to the base BL: and the solids GH and BO have also equal altitudes; consequently they are equal (Prop. 56). But it has been shown that the prism ACE is equal to the solid BO; therefore the prism ACE is likewise equal to the parallelepiped GH. Wherefore if a prism having a triangle for its base, &c.: which was to be proved.

Prop. LXII. Theor.

If a prism having any rectilineal figure for its base, and a parallelepiped be upon equal bases, and have the same altitude, they shall be equal to one another.

Let the prism ABCDEFGHIK (fig. 70) having the polygon ABCDE for its base, and the parallelepiped LM, be upon equal bases ABCDE and LN, and have equal altitudes; the prism ABCDEFGHIK is equal to the parallelepiped LM.

In the base of the prism join AC, AD, and in the opposite plane, FH, FI, dividing these planes into triangles. Describe the parallelogram OP equal to the triangle ABC (I. 42); to PQ apply the parallelogram QR equal to the triangle ACD and having the angle PQS equal to the angle UOQ (I. 44); and to RS apply the paral-

lelogram ST equal to the triangle ADE and having the angle RSW equal to PQS or UOQ: so that OQSW and UPRT are straight lines. Upon the base OT let the parallelepiped OY be constituted, having the same altitude as the prism BK, and therefore as the parallelepiped LM, and upon the bases OP, QR, ST the parallelepipeds OV, QX, SY, having the same altitude.

Because AF and CH are equal and parallel to GB (Def. 15), AF and CH are equal and parallel; therefore AC and FH are equal and parallel, and AH is a parallelogram. Also FG and GH are equal and parallel to AB and BC each to each; therefore the triangle FGH is equal (I. 8), similar (VI. Def. 1) and parallel (Prop. 26) to the triangle ABC; consequently the solid ACG is a prism (Def. 15) on the triangle ABC as its base; and it is equal to the parallelepiped OV, upon the equal base OP and having an equal altitude (Prop. 61). In the same manner it may be shown that the solids ACI and ADK are prisms on the triangular bases ACD and ADE, and that they are equal to the parallelepipeds QX and SY upon the equal bases QR and ST, and have also equal altitudes. The whole prism upon the base ABCDE is therefore equal to the solid OY. The bases LN and OT, being each equal to the base ABCDE, are equal to one another, and therefore the parallelepiped LM is equal to the parallelepiped OY. But the prism upon the base ABCDE has been shown to be equal to the solid OY; consequently it is likewise equal to the parallelepiped LM upon an equal base and having an equal altitude. Wherefore if a prism having, &c.: which was to be proved.

Cor. 1. Prisms which have equal bases and altitudes are equal to one another; for each is equal to a parallelepiped standing upon an equal base and having an equal altitude.

Cor. 2. Prisms having equal altitudes are to one another as their bases: for parallelepipeds standing upon bases respectively equal to the bases of the prisms and having an equal altitude (to which parallelepipeds, by this proposition, the prisms are respectively equal), are to another as their bases (Prop. 57), that is, as the bases of the prisms.

Cor. 3. Prisms are to one another in the compound ratio of their bases and altitudes; for parallelepipeds having bases respectively equal to the bases, and altitudes equal to the altitudes of the prisms, (to which the prisms are respectively equal,) have to one another the ratio compounded of the ratios of their bases and altitudes (Prop. 58), that is, the ratio compounded of the ratios of the bases and altitudes of the prisms.

Cor. 4. Similar prisms are to one another in the triplicate ratio of their homologous sides: for, by the last corollary, the prisms are to one another in the ratio compounded of the ratios of their bases and altitudes: the bases being similar rectilineal figures (Def. 13) are to

one another in the duplicate ratio of their homologous sides (VI. 20); and the solids being similar, their altitudes are in the simple ratio of the homologous sides: the prisms, therefore, are to one another in the ratio compounded of the duplicate ratio of two homologous sides and the simple ratio of the same sides; that is, in the triplicate ratio of the homologous sides.

Prop. LXIII. Theor.

If a pyramid be cut by a plane parallel to the base, the section will be similar to the base; and this section and the base will be to one another in the duplicate ratio of their distances from the vertex of the pyramid.

Let the pyramid ABCDEV (fig. 71) be cut by the plane FI parallel to the base AD, and let VL be the perpendicular to the base from the vertex V, meeting the plane FI in M; the section FGHIK is similar to the base ABCDE, and is to the base in the duplicate ratio of VM to VL.

Because the parallel planes AD and FI are cut by the plane AVB, the sections FG and AB are parallel (Prop. 27). For the same reason GH is parallel to BC. And because FG and GH are parallel to AB and BC, the angle FGH is equal to the angle ABC. In like manner it may be shown that the angles BCD, CDE, DEA, EAB are severally equal to the angles GHI, HIK, IKF, KFG; consequently ABCDE and FGHIK are equiangular. And because FG is parallel to AB the triangles VFG and VAB are equiangular (I. 29); and for the like reason, the triangles VGH, VBC are equiangular; consequently

$$\begin{array}{rl} FG : GV :: AB : BV & \\ \text{and } GV : GH :: BV : BC & \\ \therefore \text{ ex æquali } FG : GH :: AB : BC & (\text{V. } 22); \end{array}$$

that is, the sides about the equal angles FGH and ABC are proportionals. In like manner it may be shown that the sides about the other equal angles of ABCDE and FGHIK are proportionals; consequently the section FGHIK is similar to the base ABCDE.

And because FGHIK is similar to ABCDE, they are to one another in the duplicate ratio of FG to AB (VI. 20), that is, of VF to VA, because FG is to AB as VF is to VA. But joining AL, FM, the triangles VFM, VAL are similar, because VML is perpendicular to each of the planes AD, FI (Prop. 28); therefore VF is to VA as VM is to VL: consequently the section FGHIK is to the base ABCDE in the duplicate ratio of VM to VL.

Wherefore if a pyramid be cut, &c.: which was to be proved.

Prop. LXIV. Theor.

If two pyramids which have equal bases and altitudes be cut by planes that are parallel to the bases, and at equal distances from the vertices, the sections will be equal to one another.

Let the two pyramids ABCV and DEZ (fig. 72) which have equal bases, the triangle ABC and the quadrilateral DE, and equal altitudes VL and ZM, be cut by planes parallel to their bases, at equal distances, VN and ZO, from their vertices V and Z; the section FGH is equal to the section IK.

By the last proposition, the section FGH is to the base ABC in the duplicate ratio of VN to VL, and the section IK is to the base DE in the duplicate ratio of ZO to ZM; but VN is equal to ZO, and VL to ZM; therefore the duplicate ratio of VN to VL is the same as the duplicate ratio of ZO to ZM; and consequently FGH is to ABC as IK to DE: but ABC is equal to DE; therefore FGH is equal to IK. Wherefore if two pyramids, &c.: which was to be proved.

Prop. LXV. Theor.

A series of prisms all of the same altitude may be described about any pyramid, and another series of prisms all of the like altitude may be inscribed in the pyramid, such that the sum of the first series shall exceed the sum of the second series by a prism the base of which is the base of the pyramid, and of which the altitude is the altitude of each of the prisms: and the pyramid will be intermediate to the two series of prisms so that the sum of the first series will exceed the pyramid, and the pyramid will exceed the sum of the second series, by a solid less than the prism of which the base is the base of the pyramid, and the altitude is the altitude of each of the prisms.

Let VABC (fig. 73) be any pyramid of which the vertex is V; let Vc be divided into any number of equal parts, in the points c_1, c_2, c_3, &c., and through the points of division c_1, c_2, c_3, &c., let planes pass parallel to the base ABC, meeting the prism in the sections $a_1b_1c_1$, $a_2b_2c_2$, $a_3b_3c_3$, &c., which will be at equal distances from one another (Prop. 31), and will be all similar to the base ABC and to one another.

From the point A draw in the plane AVC the straight line Ad_1 parallel to Cc_1, meeting C$_1a_1$ produced in d_1; in like manner, from the point B draw in the plane BVC, Be_1 parallel to Cc_1, meeting c_1b_1 produced in e_1; and join d_1e_1: then it is evident that the figure d_1Ce_1 is a prism (Def. 15) and circumscribes a portion of the pyramid VABC. By similar constructions the other circumscribing prisms $d_2c_1e_2$, $d_3c_2e_3$, &c., are described. Produce d_2a_1 and e_2b_1, which are in the planes of the triangles VAC, VBC, to meet AC

and BC in f_1 and i_1; and join $f_1 i_1$; then $a_1 C b_1$ is a prism inscribed in a portion of the pyramid. In a similar manner the other inscribed prisms $a_2 c_1 b_2$, $a_3 c_2 b_3$, &c. are described.

The sum of the circumscribed prisms $d_5 c_4 e_5$, $d_4 c_3 e_4$, &c. exceeds the sum of the inscribed prisms $a_4 c_3 b_4$, $a_3 c_2 b_3$, &c., by the prism $d_1 C e_1$, of which the base is ABC, the base of the pyramid, and altitude, the altitude of each of the prisms.

Because the perpendicular distances between the planes intersecting the pyramid are all equal, the altitudes of all the circumscribed and inscribed prisms are equal: therefore the circumscribed prism $d_5 c_4 e_5$ is equal to the inscribed prism $a_4 c_3 b_4$. For the same reason the circumscribed prisms $d_4 c_3 e_4$, $d_3 c_2 e_3$, $d_2 c_1 e_3$ are severally equal to the inscribed prisms $a_3 c_2 b_3$, $a_2 c_1 b_2$, $a_1 C b_1$. Consequently the sum of the circumscribed prisms, exclusive of the prism $d_1 C e_1$, is equal to the sum of the inscribed prisms; and therefore the sum of all the circumscribed prisms exceeds the sum of all the inscribed prisms, by the prism $d_1 C e_1$, of which the base is ABC and of which the altitude is the altitude of each of the prisms.

It is evident that the pyramid VABC, being within the circumscribed prisms, and the inscribed prisms being within the pyramid, it is less than the sum of the first series of prisms, and greater than the sum of the second; and since it has been shown that the former sum exceeds the latter by the prism $d_1 C e_1$, the sum of the first series of prisms exceeds the pyramid, and the pyramid exceeds the sum of the second series of prisms, by a solid less than the prism $d_1 C e_1$.

Wherefore a series of prisms, &c.: which was to be proved.

Prop. LXVI. Theor.

A series of prisms, all of the same altitude, may be described about any pyramid, such that the sum of the prisms shall exceed the pyramid by a solid less than any given solid, however small.

Let VABC (fig. 73) be any pyramid, and S a given solid, which is equal to a prism of which the base is ABC, and altitude the perpendicular drawn from a point G in the line VC upon the plane ABC; a series of prisms of the same altitude may be circumscribed about the pyramid VABC, such that the sum of the prisms shall exceed the pyramid VABC by a solid less than the solid S, or the prism of which the base is ABC and altitude the perpendicular drawn from G upon the plane ABC.

It is evident that GC, however small, may be multiplied so as to exceed CV. Let then n denote the number of times that GC being taken, the multiple exceeds CA; and divide CV into as many equal parts as n denotes units, and let these be Cc_1, $c_1 c_2$, $c_2 c_3$, &c., each of which will be less than GC (V. Ax. 4).

Circumscribing the prisms $d_1 C e_1$, $d_2 c_1 e_2$, $d_3 c_2 e_3$, &c., as in the last

proposition, these circumscribed prisms exceed the pyramid by a solid less than the prism d_1Ce_1; but the prism d_1Ce_1 is less than the prism which has the same base ABC, and altitude the perpendicular drawn from G on the plane ABC, that is less than the solid S; with much greater reason then must the circumscribed prisms exceed the pyramid by a solid less than the solid S. Wherefore a series of prisms, &c.: which was to be proved.

Prop. LXVII. Theor.

Pyramids that have equal bases and altitudes are equal to one another.

Let VABC, ULMN (fig. 74) be two pyramids that have equal bases ABC, LMN, and equal altitudes, viz. the perpendiculars drawn from the vertices V, U, upon the planes ABC, LMN: the pyramid VABC is equal to the pyramid ULMN.

If they are not equal, one of them is the greater; let ULMN be the greater, and let it exceed the pyramid VABC by a solid S. Then a series of prisms of the like altitudes may be described about the pyramid VABC that shall exceed it by a solid less than S (Prop. 66); let these be the prisms that have for their bases ABC, $a_1b_1c_1$, $a_2b_2c_2$, &c. Whatever number of equal parts VA is divided into, divide UL into the same number in the points l_1, l_2, l_3, &c.; and through these points, let the sections $l_1n_1m_1$, $l_2n_2m_2$, $l_3n_3m_3$, &c. be made parallel to the base LNM. Then since the bases and altitudes of the pyramids are equal, and the sections $a_1b_1c_1$, $l_1m_1n_1$ are at equal distances from the vertices V, U, the equal altitudes of the pyramids being divided into the same number of equal parts, $a_1b_1c_1$ is equal to $l_1m_1n_1$ (Prop. 64); and for the same reason, the sections $a_2b_2c_2$, $a_3b_3c_3$, &c. are respectively equal to $l_2m_2n_2$, $l_3m_3n_3$, &c.; consequently the prism which stands on the base ABC, and is between the planes ABC, $a_1b_1c_1$, is equal to the prism which stands upon the base LMN, and is between the planes LMN, $l_1m_1n_1$, because they have equal altitudes (Prop. 62. Cor. 1); and for the same reason, the prisms which stand upon the bases $a_1b_1c_1$, $a_2b_2c_2$, &c., and are respectively between the planes $a_1b_1c_1$ and $a_2b_2c_2$, $a_2b_2c_2$ and $a_3b_3c_3$, &c., are severally equal to the prisms which stand upon the bases $l_1m_1n_1$, $l_2m_2n_2$, &c., and are respectively between the planes $l_1m_1n_1$ and $l_2m_2n_2$, $l_2m_2n_2$ and $l_3m_3n_3$, &c.: wherefore the sum of all the prisms described about the pyramid VABC is equal to the sum of all the prisms described about the pyramid ULMN. But, by construction, the excess of the prisms described about the pyramid VABC above the pyramid VABC is less than the solid S; and, therefore, the excess of the prisms described about the pyramid ULMN above the pyramid VABC is also less than S. But the excess of the pyramid ULMN above the pyramid VABC is, by hypothesis, equal to S; therefore the pyramid ULMN exceeds the pyra-

mid VABC more than the sum of the prisms described about the pyramid ULMN exceeds the same pyramid VABC; and consequently the pyramid ULMN is greater than the sum of the prisms described about it, which is impossible (Prop. 65). The pyramids VABC, ULMN are, therefore, not unequal, that is they are equals to one another. Wherefore, pyramids that have, &c.: which was to be proved.

Prop. LXVIII. Theor.

Every prism having a triangular base may be divided into three pyramids that have triangular bases, and that are equal to one another.

Let ABCDEF (fig. 75) be a prism of which the base is the triangle ABC, and the triangle DEF is the side opposite to the base: the prism ABCDEF may be divided into three pyramids that have triangular bases, and that are equal to one another.

Join AE, EC, CD: the planes DEC, AEC divide the prism into three pyramids having triangular bases; and these three pyramids are equal to one another.

Because ABED is a parallelogram, of which AE is the diameter, the triangle ADE is equal to the triangle ABE (I. 34): therefore the pyramid of which the base is the triangle ADE, and vertex the point C is equal to the pyramid of which the base is the triangle ABE and vertex the point C (Prop. 67). But the pyramid of which the base is the triangle ABE, and vertex the point C, is the same solid as the pyramid of which the base is the triangle ABC, and vertex the point E; and this last pyramid is equal to pyramid of which the base is the triangle DEF and vertex the point C, these pyramids having equal bases ABC, DEF (Def. 15), and equal altitudes, the perpendiculars from E and C on the parallel planes ABC and DEF (Props. 28. 30). Therefore the three pyramids CEAD, EABC, CEDF are equal to one another. But the pyramids CEAD, EABC, CEDF make up the whole prism ABCDEF; therefore the prism ABCDEF is divided into three equal pyramids having triangular bases. Wherefore, every prism, &c.: which was to be proved.

Cor. 1. From this it is manifest that every pyramid is a third part of a prism which has the same base, and is of an equal altitude with it; for if the base of the prism be any other figure than a triangle, it may be divided into prisms having triangular bases.

Cor. 2. Pyramids having equal altitudes are to one another as their bases; because the prisms upon the same bases, and of the same altitude as the pyramids, are to one another as their bases (Prop. 62. Cor. 2).

Cor. 3. All pyramids are to one another in the compound ratio of their bases and altitudes; because the prisms having the same bases and having the same altitudes as the pyramids, and of which

these are the third parts, are to one another in the compound ratio of their bases and altitudes (Prop. 62. Cor. 3).

Cor. 4. Similar pyramids are to one another in the triplicate ratio, or as the cubes, of their homologous sides; because the similar prisms upon the same bases and having the same altitudes as the pyramids, and of which these are the third parts, are to one another in the triplicate ratio of the homologous sides of their bases (Prop. 62. Cor. 4).

Prop. LXIX. Theor.

If from any point in the circumference of the base of a cylinder, a straight line be drawn perpendicular to the plane of the base, it will be wholly in the cylindric surface.

Let ABCD (fig. 76) be a cylinder, of which the base is the circle AEB, DFC the circle opposite to the base, and GH the axis; from E, any point in the circumference AEB, let EF be drawn perpendicular to the plane of the circle AEB; the straight line EF is in the superficies of the cylinder.

Let AGHD be the rectangle by the revolution of which the cylinder ABCD is described (Def. 23); then GH being at right angles to GA, the straight line which by its revolution describes the circle AEB, it is at right angles to all the straight lines in the plane of that circle which meet it in G, and it is therefore at right angles to the plane of the circle AEB; consequently AD, which is parallel to HG, is also at right angles to the plane of the circle AEB (Prop. 18) in every position of the rectangle AGHD in its revolution about HG. When, therefore, in the revolution of the rectangle AGHD, the straight line GA coincides with GE, AD being at right angles to the plane of the circle AEB will coincide with EF, which is at right angles to the same plane, because from the same point in a plane, there can be only one perpendicular to the plane upon the same side of it (Prop. 8). But AD is always in the superficies of the cylinder, for it describes that superficies; therefore EF is also in the superficies of the cylinder ABCD. Wherefore, if from any point, &c.: which was to be proved.

Prop. LXX. Theor.

A cylinder and a parallelepiped having equal bases and altitudes are equal to one another.

Let ABCD (fig. 77) be a cylinder, and EF a parallelepiped, having equal bases, viz. the circle AGB and the parallelogram EH, and having equal altitudes; the cylinder ABCD is equal to the parallelepiped EF.

If not, let them be unequal; and first, let the cylinder ABCD be

less than the parallelepiped EF; and let it be equal to the parallelepiped EQ, a part of EF cut off by the plane PQ parallel to NF. In the circle AGB inscribe the polygon AGKBLM that shall differ from the circle by a space less than the parallelogram PH (Supp. Prop. 8. Cor. 1), and cut off from the parallelogram EH, a part OR equal to the polygon AGKBLM: the point R will fall between P and N; because the parallelogram EH is equal to the circle AGB. On the polygon AGKBLM let a right prism AGBCD, of the same altitude as the cylinder, be constituted, by drawing from A, G, K, B, L, M, straight lines perpendicular to the plane AGB (Prop. 7), which prism will be less than the cylinder because it is within it (Prop. 69); and if through the point R a plane RS parallel to NF be made to pass, it will cut off the parallelepiped ES equal to the prism AGBC, because its base is equal to that of the prism and its altitude is the same (Prop. 62). But the prism AGBCD is less than the cylinder ABCD, and the cylinder ABCD is equal to the parallelepiped EQ, by hypothesis; therefore ES is less than EQ, and it is also greater, which is impossible. The cylinder ABCD, therefore, is not less than the parallelepiped EF. In a similar manner it may be shown that the cylinder is not greater than the parallelepiped EF. They therefore are equal. Wherefore, a cylinder and a parallelepiped, &c.: which was to be proved.

Cor. 1. A cylinder and a prism having equal bases and altitudes are equal to one another; for each of them is equal to a parallelepiped of an equal base and altitude (Props. 62, 70).

Cor. 2. Cylinders having equal altitudes are to one another as their bases; for the parallelepipeds to which they are respectively equal are to one another as their bases (Prop. 57), that is as the bases of the cylinders. Therefore (Supp. Prop. 10) cylinders having equal altitudes are to one another as the squares of the diameters of their bases.

Cor. 3. For a similar reason (Prop. 58) cylinders are to one another in the ratio compounded of the ratios of their bases and altitudes. They are therefore to one another in the compound ratio of their altitudes and the squares of the diameters of their bases.

Cor. 4. Again, for a like reason (Prop. 59. Cor.), similar cylinders are to one another as the cubes of the diameters of their bases, or as the cubes of their altitudes.

Prop. LXXI. Theor.

If a cone be cut by a plane parallel to the base, the section will be a circle; and this section and the base will be to one another in the duplicate ratio of their distances from the vertex of the cone.

Let VAB (fig. 78) be any cone of which VC is the axis, and VD a perpendicular from V on the base AHB; and let the cone VAB

be cut by a plane EKF parallel to the base AHB: EKF is a circle, and EKF is to AHB as the square of VL to the square of VD.

Let VAB be a section of the cone passing through the axis VC and VD, the perpendicular from V on the base AHB, cutting the plane EKF in EGLF; and let VHC be a section of the cone by any other plane passing through VC, and cutting EKF in KG.

Then since EF and AB are parallel (Prop. 27) and likewise KG and HC, the triangles VAC and VEG, VHC and VKG, VCD and VGL, are similar.

Consequently AC : CV :: EG : GV
CV : CH :: GV : GK
therefore, ex æquali AC : CH :: EG : GK (V. 22);
but AC is equal to CH, therefore EG is equal to GK (V. A.).

In like manner it may be shown that any other straight line drawn from G to the circumference of the section EKF is equal to EG; therefore the section EKF is a circle, and G is its centre.

Since circles are to each other as the squares of their diameters (Supp. Prop. 10), and therefore as the squares of their radii (V. 15), the circle EKF is to the circle AHB as the square of EG to the square of AC; but by similar triangles
EG : GV :: AC : CV
GV : VL :: CV : VD
therefore, ex æquali EG : VL :: AC : VD (V. 22);
therefore, alternando EG : AC :: VL : VD (V. 16);
and therefore the square of EG is to the square of AC, as the square of VL to the square of VD (VI. 22). Consequently the circle EKF is to the circle AHB, as the square of VL to the square of VD. Wherefore if a cone be cut by a plane, &c.: which was to be proved.

Prop. LXXII. Theor.

If a right cone and a cylinder have the same base and the same altitude, the cone is the third part of the cylinder.

Def. *A right cone* is a cone of which the axis is at right angles to the plane of the base, and which therefore may be described by the revolution of a right-angled triangle about one of its sides containing the right angle, which side remains fixed.

Let the cone ABCD (fig. 79) and the cylinder BFKG have the same base, the circle BCD, and the same altitude, the perpendicular from the point A upon the plane BCD, the cone ABCD is the third part of the cylinder BFKG.

For if not, let the cone ABCD be the third part of another cylinder LMNO, having its altitude equal to that of the cylinder BFKG, but its base LIM not equal to the base BCD; and first, let BCD be greater than LIM; therefore the cylinder LMNO less

than the cylinder BFKG; and consequently the cone ABCD less than the third part of the cylinder BFKG.

Then, because the circle BCD is greater than the circle LIM, a polygon may be inscribed in BCD, that shall differ from it less than LIM does (Supp. Prop. 8. Cor. 1), and which, therefore, will be greater than LIM. Let this be the polygon BECFD; and upon BECFD, let there be constituted the pyramid ABECFD, and the prism BCFKHG.

Because the polygon BECFD is greater than the circle LIM, the prism BCFKHG is greater than the cylinder LMNO, for their altitudes are equal and the prism has the greater base (Prop. 70. Cor. 1). But the pyramid ABECFD is the third part of the prism BCFKHG (Prop. 68. Cor. 1), and therefore it is greater than the third part of the cylinder LMNO. Now the cone ABECFD is, by hypothesis, the third part of the cylinder LMNO; therefore the pyramid ABECFD is greater than the cone ABCD; and it is also less, because it is inscribed in the cone; which is impossible. Therefore the cone ABCD is not less than the third part of the cylinder BFKG.

In like manner, by circumscribing a polygon about the circle BCD, it may be shown that the cone ABCD is not greater than the third part of the cylinder BFKG: therefore the cone ABCD is equal to the third part of the cylinder BFKG. Wherefore, if a cone and a cylinder, &c.: which was to be proved.

Prop. LXXIII. Theor.

A series of cylinders all of the same altitude may be described about any cone, and another series of cylinders, all of the like altitude, may be inscribed in the cone, such that the sum of either series shall differ from the cone by a solid less than any given solid, however small.

Let VAB (fig. 80) be any cone; then a series of cylinders all of the same altitude may be circumscribed about VAB, and another series of cylinders all of the like altitude may be inscribed in VAB, such that the sum of either series shall differ from VAB by a solid less than any given solid S, however small.

Let VACB be a section of the cone by a plane passing through the axis VC perpendicular to the base. Divide the axis of the cone, VC into any number of equal parts in the points c_1, c_2, c_3, c_4; through these points draw straight lines parallel to AB the diameter of the base; and complete the series of circumscribed and inscribed rectangles in the triangle VAB, and through C, c_1, c_2, &c. draw Ce_1, $i_1c_1e_2$, $i_2c_2e_3$, &c. perpendicular to the bases of the circumscribed and inscribed rectangles: then if the circumscribed rectangles revolve about the sides Ce_1, c_1e_2, c_2e_3, &c., and the inscribed rectangles revolve about the sides c_1i_1, c_2i_2, c_3i_3, &c., it is evident that they will

describe a series of cylinders circumscribed about the cone VAB, and another series of cylinders inscribed in it.

As in the case of the pyramid, and the inscribed and circumscribed prisms (Prop. 65), it is evident that the sum of the series of circumscribed cylinders will exceed the sum of the series of inscribed cylinders by the cylinder described by the rectangle Ch_1. Since the cone VAB is intermediate to the two series of cylinders, it will differ from the sum of either series by a solid less than the cylinder described by the rectangle Ch_1.

Let the solid S be equal to a cylinder of which the base is the base of the cone, and the altitude is a. Then, if the axis VC be divided into such a number of equal parts that Ce_1 is less than a, it is evident that the sum of the series of circumscribed cylinders will exceed the sum of the series of inscribed cylinders by a solid (the cylinder described by the rectangle Ch_1), less than the solid S: and consequently that the sum of either series will differ from the cone VAB, which is intermediate to the two, by a solid less than the given solid S, however small this may be. Wherefore a series of cylinders, &c.: which was to be proved.

Cor. 1. Since the proposition applies to any cone cut from the cone VAB, by a plane parallel to the base, it applies also to the remaining portion of the cone, towards the base, called *a frustum of the cone*; that is, a series of cylinders all having the same altitude may be described about any *frustum of a cone*, and another series of cylinders all of the like altitude may be inscribed in the frustum, such that the sum of either series shall differ from the frustum by a solid less than any given solid, however small.

Cor. 2. In like manner, as was shown in Prop. 67, that pyramids which have equal bases and altitudes are equal to one another, it may be demonstrated from this proposition that cones having equal bases and altitudes are equal to one another.

Cor. 3. Any cone is a third part of a cylinder of the same base and altitude: for it is equal to a right cone of the same base and altitude.

Cor. 4. Cones are to one another in the compound ratio of their bases and altitudes; for the cylinders of which they are equal parts are in that ratio (Prop. 70. Cor. 3).

Cor. 5. For the like reason (Prop. 70. Cor. 4) similar cones are to one another as the cubes of the diameters of their bases, or as the cubes of their altitudes.

Prop. LXXIV. Theor.

If a sphere be cut by a plane, the section will be a circle.

Let ABC (fig. 81) be a sphere, and let it be cut by the plane BDC: the section BDC is a circle.

From E, the centre of the sphere, draw EF perpendicular to the plane BDC (Prop. 5); through the point F draw the straight line BFC; in the line of section BDC of the plane with the sphere take any point D; and join DF, DE, BE.

Because EF is perpendicular to the plane BDC, each of the angles EFD, EFB is a right angle; and because DE is equal to BE (Defs. 16. 18), the square of DE is equal to the square of BE; but the square of DE is equal to the squares of DF and FE (I. 47), and the square of BE is equal to the squares of BF and FE; therefore the squares of DF and FE are equal to the squares of BF and FE. From each of these equals take away the common square of FE, and there remains the square of DF equal to the square of BF; consequently DF is equal to BF. In like manner it may be shown, that any straight line drawn from a point in the line of section BDC, to the point F is equal to BF or DF: therefore the section BDC is a circle (I. Def. 15); and the point F is its centre. Wherefore if a sphere be cut by a plane, &c.: which was to be proved.

PROP. LXXV. THEOR.

A series of cylinders all of the same altitude may be described about a hemisphere, and another series of cylinders all of the like altitude may be inscribed in the hemisphere, such that the sum of either series shall differ from the hemisphere by a solid less than any given solid, however small.

Let ADB (fig. 82) be a semicircle of which the diameter is AB, the centre C, and CD a radius at right angles to AB: if either of the sectors ACD, BCD revolve about DC, it will describe a hemisphere, having C for its centre. Also, if DC be divided into any number of equal parts in the points c_1, c_2, c_3, c_4; through these points straight lines be drawn parallel to AB, and to the semicircle, circumscribed and inscribed rectangles be drawn, as in the figure; if the figure revolve about CD, either half of these rectangles will describe series of cylinders described about and in the hemisphere described by either of the sectors ACD, BCD: then CD may be so divided that the sum of either series of cylinders shall differ from the hemisphere by a solid less than any given solid S, however small.

It is evident that here, as in the case of the cone (Prop. 73), the sum of the series of circumscribed cylinders exceeds the sum of the series of inscribed cylinders by the cylinder described by the rectangle Ce_1 or Cf_1. If then the given solid S be equal to a cylinder of which the base is the circle described by CA or CB, and the altitude is a; and CD be divided into such a number of equal parts that Cc_1 is less than a; it is evident that the sum of the series of circumscribed cylinders will exceed the sum of the series of inscribed cylinders by a solid (the cylinder described by the rectangle Ce_1) less than the solid

S; and consequently that the sum of either series will differ from the hemisphere described by the sector ACD, which is intermediate to the two, by a solid less than the given solid S, however small this may be. Wherefore a series of cylinders, &c.: which was to be proved.

Cor. The proposition is true for any segment of the hemisphere; for the demonstration will be the same, whether it be CD which is divided into equal parts, and rectangles are described about and in the semicircle, or that it is a portion of CD which is so divided, and rectangles are described about and in the segment cut off.

Prop. LXXVI. Theor.

If a hemisphere and a cone have equal bases and altitudes, a series of cylinders may be inscribed in the hemisphere, and another series may be described about the cone, having all the same altitudes, such that their sum shall differ from the sum of the hemisphere and the cone by a solid less than any given solid, however small.

Let ADB (fig. 83) be a semicircle of which the diameter is AB, the centre C, and CD a radius at right angles to AB; let CE and CF be squares described on CD; and join CE, CF: then if the figure revolve about CD, either of the sectors ACD, BCD will describe a hemisphere having C for its centre, and either of the triangles CDE, CDF will describe a cone having its vertex at C, and having for its base the circle described by DE or DF, equal to that described by CA or CB, which is the base of the hemisphere. Let S be any given solid. A series of cylinders may be inscribed in the hemisphere, and another series described about the cone, such their sum shall differ from the sum of the hemisphere and the cone, by a solid less than S, however small S may be.

Let the given solid S be equal to a cylinder of which the base is equal to the base of the hemisphere or of the cone, and the altitude is a. Divide CD into such a number of equal parts in c_1, c_2, c_3, &c. that Cc_1 is less than a; through c_1, c_2, c_3, &c. draw straight lines parallel to AB or EF; and complete the rectangles inscribed in the semicircle, as in the last proposition, and the rectangles described about the triangle ECF as in Proposition 73: then if the figure revolve about CD, the rectangle Ca_1, c_1a_2, c_2a_3, &c. will describe cylinders in the hemisphere ADB, and the rectangles c_1k_1, c_2k_2, c_3k_3, &c. will describe cylinders about the cone ECF.

Since, by the last proposition, the hemisphere ADB exceeds the sum of the series of cylinders inscribed in it, by a solid less than S, let this excess be a solid T which is less than S; so that the sum of the series of cylinders inscribed in the hemisphere together with the solid T is equal to the hemisphere. And since, by Proposition 73, the sum of the series of cylinders described about the cone exceeds

the cone by a solid likewise less than S, let this excess be a solid V which is less than S; so that the sum of the series of cylinders described about the cone is equal to the cone together with the solid V. Hence, adding equals to equals, the sum of the two series of cylinders, the one inscribed in the hemisphere, and the other described about the cone, together with the solid T, is equal to the sum of the hemisphere and the cone together with the solid V. From each of these equals take away the solid T, then the sum of the two series of cylinders will differ from the hemisphere and cone by a solid which is the difference of the solids T and V; and since T and V are each less than S, their difference must be less than S: consequently the sum of the two series of cylinders, the one inscribed in the hemisphere, and the other described about the cone, differ from the sum of the hemisphere and the cone by a solid less than the solid S, however small S may be. Which was to be proved.

Cor. The proposition is true for any segment of the sphere, which would be described by the revolution of the figure DIG about DC, and any portion of the cone, which would be described by the revolution of the figure DFHG, likewise about DC. For the demonstration in the proposition applies equally to the segment of the sphere and the portion of the cone, making use of the corollaries to Propositions 75 and 73, instead of the propositions themselves.

Prop. LXXVII. Theor.

The same things being supposed as in the last proposition, the sum of the two series of cylinders, the one inscribed in the hemisphere, and the other described about the cone, is equal to a cylinder having the same base and altitude as the hemisphere, or the cylinder circumscribing the hemisphere.

Let the figure be constructed as in the last proposition (fig. 83), and let it revolve about CD; the sum of the two series of cylinders, the one described by the rectangles Ca_1, c_1a_2, c_2a_3, &c., and which are inscribed in the hemisphere ADB, and the other described by the rectangles c_1k_1, c_2k_2, c_3k_3, &c., and which are described about the cone ECF, is equal to the cylinder described by the square DA or the square DB, that is, to the cylinder circumscribing the hemisphere.

Let a straight line passing through one of the points of division in CD, cut the semicircle in I, CF in H, and FB in K; join CI. Then because CGI is a right angle, the circles described with the distances CG and GI are equal to the circle described with the distance CI or GK (Supp. Prop. 10. Cor. 2); now CG is equal to GH, because GH is parallel to DF, and CD is equal to DF; therefore the circles described with the distances GH and GI are equal to the circle described with the distance GK, that is, the circles described

by the revolution of GH and GI about the axis CD are equal to the circle described by the revolution of GK about the same axis. And since cylinders having the same altitude are to one another as their bases (Prop. 70. Cor. 2), the cylinder described by the rectangle GM, is to the cylinder described by GO, as the circle described by GH is to the circle described by GK; and the cylinder described by GN is to the cylinder described by GO, as the circle described by GI is to the circle described by GK; therefore the cylinders described by GM and GN together, are to the cylinder described by GO, as the circles described by GH and GI together, are to the circle described by GK (V. 24): but the circles described by GH and GI are together equal to the circle described by GK; therefore the cylinders described by GM and GN are together equal to the cylinder described by GO (V. A). In like manner it may be shown, that each of the cylinders described about the cone, together with the corresponding cylinder inscribed in the hemisphere, is equal to a cylinder of the same altitude, and having its base equal to the base of the cone or of the hemisphere; therefore the sum of the two series of cylinders inscribed in the hemisphere and described about the cone is equal to the sum of all the cylinders having the same altitude, and having their bases equal to the base of the cone or of the hemisphere, that is, to the cylinder described by the square DB, or the cylinder circumscribing the hemisphere. Which was to be proved.

Cor. The sum of the two series of cylinders, the one inscribed in a segment of the sphere, and the other in the corresponding frustum of the cone, is equal to the cylinder having the same base as the cone, and the same altitude as the frustum. This is shown in the demonstration of the proposition.

Prop. LXXVIII. Theor.

Every sphere is two-thirds of the circumscribing cylinder.

Let the figure be constructed as in the last two propositions (fig. 83); then the hemisphere ADB, described by the sector DCB, and the cone ECF, described by the triangle FDC, are together equal to the cylinder ABFE, described by the square DB: for if they be not equal, let them be unequal, and let the hemisphere ADB, and cone ECF together, differ from the cylinder ABFE by a solid S. Then however small S may be, a series of cylinders may be inscribed in the hemisphere, and described about the cone, such that their sum shall differ from the sum of the hemisphere ADB and the cone ECF by a solid less than S (Prop. 76); but the sum of the series of cylinders inscribed in the hemisphere ADB, and described about the cone ECF, is equal to the cylinder ABFE circumscribing the hemisphere (Prop. 77): therefore the sum of the hemisphere ADB and the cone ECF differs from the cylinder ABFE circumscribing the

hemisphere, by a solid less than S; but by supposition it differs by the solid S, which is absurd. Therefore the hemisphere ADB and the cone ECF together are not unequal to the circumscribing cylinder ABFE, that is, they are equal to that cylinder. And because the cone ECF is the third part of the cylinder ABFE, the hemisphere ADB is two-thirds of the circumscribing cylinder ABFE; and consequently the whole sphere is two-thirds of the cylinder described by twice the rectangle DB, that is, two-thirds of its circumscribing cylinder. Which was to be proved.

Cor. 1. A cone, a hemisphere, and a cylinder, all having the same base and the same altitude, are to one another as the numbers 1, 2 and 3.

Cor. 2. Spheres are to one another as the cubes of their diameters; for they are to one another as their circumscribing cylinders (V. C), and these being similar are to one another as the cubes of their altitudes, that is, as the cubes of the diameters of the spheres.

Cor. 3. From the corollaries to propositions 76 and 77, it follows, as in the above demonstration, that the segment of the sphere described by the revolution of DIG about DC, and the frustum of the cone described by DFHG, are together equal to the cylinder described by DFKG; and that the zone of the sphere described by GIBC and the cone described by GHC, are together equal to the cylinder described by GKBC.

THEOREMS TO BE DEMONSTRATED.

1. Three straight lines, meeting in a point, but which are not all in one plane, being given, a parallelepiped three of whose edges are equal to these straight lines may be constructed.

2. Three straight lines, not parallel to a plane, and no two of which are in the same plane, being given, a parallelepiped three of whose edges are in these straight lines may always be constructed.

3. If planes pass through each of the diagonals of two opposite parallelograms of a parallelepiped, they will divide it into four equal prisms.

4. If two opposite solid angles of a parallelepiped be joined by a straight line, and any other two opposite solid angles be likewise joined, these two straight lines will intersect.

5. The intersections of planes perpendicular to the edges of any triangular pyramid at their middle points all intersect in one point; and this point is the centre of a sphere the surface of which passes through the vertices of the four solid angles of the pyramid.

6. If a cylinder be cut by a plane parallel to the base, the section is a circle equal to the base.

7. If a cone be cut by a plane passing through its axis, perpendicular to the base, and planes touch the cone in the opposite lines

of section, then if the cone be cut by a plane making the same angle with the tangent plane on one side, which the base makes with the tangent plane on the other, the section (which is called *subcontrary*) will be a circle; and this section will be to the base in the duplicate ratio of its distance from the vertex, to the altitude of the cone.

8. Show that a sphere may be described about any cone.

END OF PART III.

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